Semi-classical analysis

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Chapter 1

Introduction

Let $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates $(x^1, \ldots, x^n, t)$. Let

$$P = P\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$$

be a k-th order linear partial differential operator. Suppose that we want to solve the partial differential equation

$$Pu = 0$$

with initial conditions

$$u(x, 0) = \delta_0(x), \quad \frac{\partial^i}{\partial t^i} u(x, 0) = 0, \quad i = 1, \ldots, k-1,$$

where δ_0 is the Dirac delta function.

Let ρ be a C^∞ function of x of compact support which is identically one near the origin. We can write

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}^n} e^{ix \cdot \xi} d\xi.$$

Let us introduce polar coordinates in ξ space:

$$\xi = \omega \cdot r, \quad \|\omega\| = 1, \quad r = \|\xi\|$$

so we can rewrite the above expression as

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}_+} \int_{S^{n-1}} e^{i(x \cdot \omega) r} r^{n-1} dr d\omega$$

where $d\omega$ is the measure on the unit sphere S^{n-1}.

Passing the differential operator under the integrals shows that we are interested in solving the partial differential equation $Pu = 0$ with the initial conditions

$$u(x,0) = \rho(x)e^{i(x\cdot\omega)r}r^{n-1}, \quad \frac{\partial^i}{\partial t^i}u(x,0) = 0, \quad i = 1,\ldots,k-1.$$

1.1 The problem.

More generally, set

$$r = \hbar^{-1}$$

and let

$$\psi \in C^\infty(\mathbb{R}^n).$$

We look for solutions of the partial differential equation with initial conditions

$$Pu(x,t) = 0, \quad u(x,0) = \rho(x)e^{i\frac{\psi(x)}{\hbar}}\hbar^{-\ell}, \quad \frac{\partial^i}{\partial t^i}u(x,0) = 0, \quad i = 1,\ldots,k-1. \tag{1.1}$$

1.2 The eikonal equation.

Look for solutions of (1.1) of the form

$$u(x,t) = a(x,t,\hbar)e^{i\phi(x,t)/\hbar} \tag{1.2}$$

where

$$a(x,t,\hbar) = \hbar^{-\ell}\sum_{i=0}^{\infty} a_i(x,t)\hbar^i. \tag{1.3}$$

1.2.1 The principal symbol.

Define the **principal symbol** $H(x,t,\xi,\tau)$ of the differential operator P by

$$\hbar^k e^{-i\frac{x.\xi+t\tau}{\hbar}}Pe^{i\frac{x.\xi+t\tau}{\hbar}} = H(x,t,\xi,\tau) + O(\hbar). \tag{1.4}$$

We think of H as a function on $T^*\mathbb{R}^{n+1}$.

If we apply P to $u(x,t) = a(x,t,\hbar)e^{i\phi(x,t)/\hbar}$, then the term of degree $\hbar^{-k}$ is obtained by applying all the differentiations to $e^{i\phi(x,t)/\hbar}$. In other words,

$$\hbar^k e^{-i\phi/\hbar}Pa(x,t)e^{i\phi/\hbar} = H\left(x,t,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial t}\right)a(x,t) + O(\hbar). \tag{1.5}$$

So as a first step we must solve the first order non-linear partial differential equation

$$H\left(x, t, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t}\right) = 0 \tag{1.6}$$

for ϕ. Equation (1.6) is known as the **eikonal equation** and a solution ϕ to (1.6) is called an **eikonal** . The Greek word eikona $\epsilon\iota\kappa\omega\nu\alpha$ means image.

1.2.2 Hyperbolicity.

For all (x, t, ξ) the function

$$\tau \mapsto H(x, t, \xi, \tau)$$

is a polynomial of degree (at most) k in τ. We say that P is **hyperbolic** if this polynomial has k distinct real roots

$$\tau_i = \tau_i(x, t, \xi).$$

These are then smooth functions of (x, t, ξ).

We assume from now on that P is hyperbolic. For each $i = 1, \ldots, k$ let

$$\Sigma_i \subset T^*\mathbb{R}^{n+1}$$

be defined by

$$\Sigma_i = \{(x, 0, \xi, \tau) | \xi = d_x\psi, \ \tau = \tau_i(x, 0, \xi)\} \tag{1.7}$$

where ψ is the function occurring in the initial conditions in (1.1). The classical method for solving (1.6) is to reduce it to solving a system of ordinary differential equations with initial conditions given by (1.7). We recall the method:

1.2.3 The canonical one form on the cotangent bundle.

If X is a differentiable manifold, then its cotangent bundle T^*X carries a **canonical one form** $\alpha = \alpha_X$ defined as follows: Let

$$\pi : T^*X \to X$$

be the projection sending any covector $p \in T_x^*X$ to its base point x. If $v \in T_p(T^*X)$ is a tangent vector to T^*X at p, then

$$d\pi_p v$$

is a tangent vector to X at x. In other words, $d\pi_p v \in T_x X$. But $p \in T_x^* X$ is a linear function on $T_x X$, and so we can evaluate p on $d\pi_p v$. The canonical linear differential form α is defined by

$$\langle \alpha_p, v \rangle := \langle p, d\pi_p v \rangle \quad \text{if} \quad v \in T_p(T^*X). \tag{1.8}$$

For example, if our manifold is $\mathbb{R}^{n+1}$ as above, so that we have coordinates (x, t, ξ, τ) on $T^*\mathbb{R}^{n+1}$ the canonical one form is given in these coordinates by

$$\alpha = \xi \cdot dx + \tau dt = \xi_1 dx^1 + \cdots \xi_n dx^n + \tau dt. \tag{1.9}$$

1.2.4 The canonical two form on the cotangent bundle.

This is defined as

$$\omega_X = -d\alpha_X. \tag{1.10}$$

Let $q^1, \ldots, q^n$ be local coordinates on X. Then $dq^1, \ldots, dq^n$ are differential forms which give a basis of $T_x^* X$ at each x in the coordinate neighborhood U. In other words, the most general element of $T_x^* X$ can be written as $p_1(dq^1)_x + \cdots + p_n(dq^n)_x$. Thus $q^1, \ldots, q^n, p_1, \ldots, p_n$ are local coordinates on

$$\pi^{-1}U \subset T^*X.$$

In terms of these coordinates the canonical one-form is given by

$$\alpha = p \cdot dq = p_1 dq^1 + \cdots + p_n dq^n$$

Hence the canonical two-form has the local expression

$$\omega = dq \wedge \cdot dp = dq^1 \wedge dp_1 + \cdots + dq^n \wedge dp_n. \tag{1.11}$$

The form ω is closed and is of maximal rank, i.e., ω defines an isomorphism between the tangent space and the cotangent space at every point of T^*X.

1.2.5 Symplectic manifolds.

A two form which is closed and is of maximal rank is called **symplectic**. A manifold M equipped with a symplectic form is called a **symplectic manifold**. We shall study some of the basic geometry of symplectic manifolds in Chapter 2. But here are some elementary notions which follow directly from the definitions: A diffeomorphism $f : M \to M$ is called a **symplectomorphism** if $f^*\omega = \omega$. More generally if (M, ω) and (M', ω') are symplectic manifolds then a diffeomorphism

$$f : M \to M'$$

is called a symplectomorphism if

$$f^*\omega' = \omega.$$

If v is a vector field on M, then the general formula for the Lie derivative of a differential form Ω with respect to v is given by

$$D_v\Omega = i(v)d\Omega + di(v)\Omega.$$

This is known as Weil's identity. See (9.2) in Chapter 9 below. If we take Ω to be a symplectic form ω , so that $d\omega = 0$, this becomes

$$D_v\omega = di(v)\omega.$$

So the flow $t \mapsto \exp tv$ generated by v consists of symplectomorphisms if and only if

$$di(v)\omega = 0.$$

1.2.6 Hamiltonian vector fields.

In particular, if H is a function on a symplectic manifold M, then the **Hamiltonian vector field** v_H associated to H and defined by

$$i(v_H)\omega = dH \tag{1.12}$$

satisfies

$$(\exp tv_H)^*\omega = \omega.$$

Also

$$D_{v_H}H = i(v_H)dH = i(v_H)i(v_H)\omega = \omega(v_H, v_H) = 0.$$

Thus

$$(\exp tv_H)^*H = H. \tag{1.13}$$

So the flow $\exp tv_H$ preserves the level sets of H. In particular, it carries the zero level set - the set $H = 0$ - into itself.

1.2.7 Isotropic submanifolds.

A submanifold Y of a symplectic manifold is called **isotropic** if the restriction of the symplectic form ω to Y is zero. So if

$$\iota_Y : Y \to M$$

denotes the injection of Y as a submanifold of M, then the condition for Y to be isotropic is

$$\iota_Y^* \omega = 0$$

where ω is the symplectic form of M.

For example, consider the submanifold Σ_i of $T^*(\mathbb{R}^{n+1})$ defined by (1.7). According to (1.9), the restriction of $\alpha_{\mathbb{R}^{n+1}}$ to Σ_i is given by

$$\frac{\partial \psi}{\partial x_1} dx_1 + \cdots \frac{\partial \psi}{\partial x_n} dx_n = d_x \psi$$

since $t \equiv 0$ on Σ_i. So

$$\iota_{\Sigma_i}^* \omega_{\mathbb{R}^{n+1}} = -d_x d_x \psi = 0$$

and hence Σ_i is isotropic.

Let H be a smooth function on a symplectic manifold M and let Y be an isotropic submanifold of M contained in a level set of H. For example, suppose that

$$H_{|Y} \equiv 0. \tag{1.14}$$

Consider the submanifold of M swept out by Y under the flow $\exp tv_H$. More precisely suppose that

- v_H is transverse to Y in the sense that for every $y \in Y$, the tangent vector $v_H(y)$ does *not* belong to $T_y Y$ and
- there exists an open interval I about 0 in $\mathbb{R}$ such that $\exp tv_H(y)$ is defined for all $t \in I$ and $y \in Y$.

We then get a map

$$j : Y \times I \to M, \quad j(y,t) := \exp tv_H(y)$$

which allows us to realize $Y \times I$ as a submanifold Z of M. The tangent space to Z at a point (y,t) is spanned by

$$(\exp tv_H)_* TY_y \quad \text{and} \quad v_H(\exp tv_H y)$$

and so the dimension of Z is $\dim Y + 1$.

Proposition 1 *With the above notation and hypotheses, Z is an isotropic submanifold of M.*

Proof. We need to check that the form ω vanishes when evaluated on

1. two vectors belonging to $(\exp tv_H)_* TY_y$ and
2. $v_H(\exp tv_H y)$ and a vector belonging to $(\exp tv_H)_* TY_y$.

For the first case observe that if $w_1, w_2 \in T_yY$ then

$$\omega((\exp tv_H)_* w_1, (\exp tv_H)_* w_2) = (\exp tv_H)^* \omega(w_1, w_2) = 0$$

since

$$(\exp tv_H)^* \omega = \omega$$

and Y is isotropic. For the second case observe that $i(v_H)\omega = dH$ and so for $w \in T_yY$ we have

$$\omega(v_H(\exp tv_H y), (\exp tv_H)_* w) = dH(w) = 0$$

since H is constant on Y. □

If we consider the function H arising as the symbol of a hyperbolic equation, i.e. the function H given by (1.4), then H is a homogeneous polynomial in ξ and τ of the form $b(x, t, \xi) \prod_i (\tau - \tau_i)$, with $b \neq 0$ so

$$\frac{\partial H}{\partial \tau} \neq 0 \quad \text{along} \quad \Sigma_i.$$

But the coefficient of $\partial/\partial t$ in v_H is $\partial H/\partial \tau$. Now $t \equiv 0$ along Σ_i so v_H is transverse to Σ_i. Our transversality condition is satisfied. We can arrange that the second of our conditions, the existence of solutions for an interval I can be satisfied locally. (In fact, suitable compactness conditions that are frequently satisfied will guarantee the existence of global solutions.)

Thus, at least locally, the submanifold of $T^*\mathbb{R}^{n+1}$ swept out from Σ_i by $\exp tv_H$ is an $n+1$ dimensional isotropic submanifold.

1.2.8 Lagrangian submanifolds.

A submanifold of a symplectic manifold which is isotropic and whose dimension is one half the dimension of M is called **Lagrangian**. We shall study Lagrangian submanifolds in detail in Chapter 2. Here we shall show how they are related to our problem of solving the eikonal equation (1.6).

The submanifold Σ_i of $T^*\mathbb{R}^{n+1}$ is isotropic and of dimension n. It is transversal to v_H. Therefore the submanifold Λ_i swept out by Σ_i under $\exp tv_H$ is Lagrangian. Also, near $t = 0$ the projection

$$\pi : T^*\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

when restricted to Λ_i is (locally) a diffeomorphism. It is (locally) **horizontal** in the sense of the next section.

1.2.9 Lagrangian submanifolds of the cotangent bundle.

To say that a submanifold $\Lambda \subset T^*X$ is Lagrangian means that Λ has the same dimension as X and that the restriction to Λ of the canonical one form α_X is closed.

Suppose that Z is a submanifold of T^*X and that the restriction of $\pi : T^*X \to X$ to Z is a diffeomorphism. This means that Z is the image of a section

$$s : \; X \to T^*X.$$

Giving such a section is the same as assigning a covector at each point of X, in other words it is a linear differential form. For the purposes of the discussion we temporarily introduce a redundant notation and call the section s by the name β_s when we want to think of it as a linear differential form. We claim that

$$s^*\alpha_X = \beta_s.$$

Indeed, if $w \in T_xX$ then $d\pi_{s(x)} \circ ds_x(w) = w$ and hence

$$s^*\alpha_X(w) = \langle (\alpha_X)_{s(x)}, ds_x(w)\rangle =$$

$$= \langle s(x), d\pi_{s(x)} ds_x(w)\rangle = \langle s(x), w\rangle = \beta_s(x)(w).$$

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$. Let us suppose that X is connected and simply connected. Then $d\beta = 0$ implies that $\beta = d\phi$ where ϕ is determined up to an additive constant.

With some slight abuse of language, let us call a Lagrangian submanifold Λ of T^*X **horizontal** if the restriction of $\pi : T^*X \to X$ to Λ is a diffeomorphism. We have proved

Proposition 2 *Suppose that X is connected and simply connected. Then every horizontal Lagrangian submanifold of T^*X is given by a section $\gamma_\phi : X \to T^*X$ where γ_ϕ is of the form*

$$\gamma_\phi(x) = d\phi(x)$$

where ϕ is a smooth function determined up to an additive constant.

1.2.10 Local solution of the eikonal equation.

We have now found a local solution of the eikonal equation! Starting with the initial conditions Σ_i given by (1.7) at $t = 0$, we obtain the Lagrangian submanifold Λ_i. Locally (in x and in t near zero) the manifold Λ_i is given as the image of γ_{ϕ_i} for some function ϕ_i. The fact that Λ_i is contained in the set $H = 0$ then implies that ϕ_i is a solution of (1.6).

1.2.11 Caustics.

What can go wrong globally? One problem that might arise is with integrating the vector field v_H. As is well known, the existence theorem for non-linear ordinary differential equations is only local - solutions might "blow up" in a finite interval of time. In many applications this is not a problem because of compactness or boundedness conditions. A more serious problem - one which will be a major concern of this book - is the possibility that after some time the Lagrangian manifold is no longer horizontal.

If $\Lambda \subset T^*X$ is a Lagrangian submanifold, we say that a point $m \in \Lambda$ is a **caustic** if

$$d\pi_m T_m\Lambda \to T_xX. \quad x = \pi(m)$$

is *not* surjective. A key ingredient in what we will need to do is to describe how to choose convenient parametrizations of Lagrangian manifolds near caustics. The first person to deal with this problem (through the introduction of so-called "angle characteristics") was Hamilton (1805-1865) in a paper he communicated to Dr. Brinkley in 1823, by whom, under the title "Caustics" it was presented in 1824 to the Royal Irish Academy.

We shall deal with caustics in a more general manner, after we have introduced some categorical language.

1.3 The transport equations.

Let us return to our project of looking for solutions of the form (1.2) to the partial differential equation and initial conditions (1.1). Our first step was to find the Lagrangian manifold $\Lambda = \Lambda_\phi$ which gave us, locally, a solution of the eikonal equation (1.6). This determines the "phase function" ϕ up to an overall additive constant, and also guarantees that no matter what a_i's enter into the expression for u given by (1.2) and (1.3), we have

$$Pu = O(\hbar^{-k-\ell+1}).$$

The next step is obviously to try to choose a_0 in (1.3) such that

$$P\left(a_0 e^{i\phi(x,t)/\hbar}\right) = O(\hbar^{-k+2}).$$

In other words, we want to choose a_0 so that there are no terms of order $\hbar^{-k+1}$ in $P\left(a_0 e^{i\phi(x,t)/\hbar}\right)$. Such a term can arise from three sources:

1. We can take the terms of degree $k-1$ and apply all the differentiations to $e^{i\phi/\hbar}$ with none to a or to ϕ. We will obtain an expression C similar to the principal symbol but using the operator Q obtained from P by eliminating all terms of degree k. This expression C will then multiply a_0.

2. We can take the terms of degree k in P, apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a partial derivative of ϕ. The resulting expression B will involve the second partial derivatives of ϕ. This expression will also multiply a_0.

3. We can take the terms of degree k in P, apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a_0. So we get a first order differential operator

$$\sum_{i=1}^{n+1} A_i \frac{\partial}{\partial x_i}$$

applied to a_0. In the above formula we have set $t = x_{n+1}$ so as to write the differential operator in more symmetric form.

So the coefficient of $\hbar^{-k+1}$ in $P\left(a_0 e^{i\phi(x,t)/\hbar}\right)$ is

$$(Ra_0)\, e^{i\phi(x,t)/\hbar}$$

where R is the first order differential operator

$$R = \sum A_i \frac{\partial}{\partial x_i} + B + C.$$

We will derive the explicit expressions for the A_i, B and C below.

The strategy is then to look for solutions of the first order homogenous linear partial differential equation

$$Ra_0 = 0.$$

This is known as the **first order transport equation**.

Having found a_0, we next look for a_1 so that

$$P\left((a_0 + a_1\hbar)e^{i\phi/\hbar}\right) = O(h^{-k+3}).$$

From the above discussion it is clear that this amounts to solving an inhomogeneous linear partial differential equation of the form

$$Ra_1 = b_0$$

where b_0 is the coefficient of $\hbar^{-k+2}e^{i\phi/\hbar}$ in $P(a_0 e^{i\phi/\hbar})$ and where R is the *same operator as above*. Assuming that we can solve all these equations, we see that we have a recursive procedure involving the operator R for solving (1.1) to all orders, at least locally - up until we hit a caustic!

We will find that when we regard P as acting on $\frac{1}{2}$-densities (rather than on functions) then the operator R has an invariant (and beautiful) expression as a differential operator acting on $\frac{1}{2}$-densities on Λ, see equation (1.21) below. In fact, the differentiation part of the differential operator will be given by the vector field v_H which we know to be tangent to Λ. The differential operator on Λ will be defined even at caustics. This fact will be central in our study of global asymptotic solutions of hyperbolic equations.

In the next section we shall assume only the most elementary facts about $\frac{1}{2}$-densities - the fact that the product of two $\frac{1}{2}$-densities is a density and hence can be integrated if this product has compact support. Also that the concept of the Lie derivative of a $\frac{1}{2}$-density with respect to a vector field makes sense. If the reader is unfamiliar with these facts they can be found with many more details in Chapter 6.

1.3.1 A formula for the Lie derivative of a $\frac{1}{2}$-density.

We want to consider the following situation: H is a function on T^*X and Λ is a Lagrangian submanifold of T^*X on which $H = 0$. This implies that the corresponding Hamiltonian vector field is tangent to Λ. Indeed, for any $w \in T_z\Lambda,\ z \in \Lambda$ we have

$$\omega_X(v_H, w) = dH(w) = 0$$

since H is constant on Λ. Since Λ is Lagrangian, this implies that $v_H(z) \in T_z(\Lambda)$.

If τ is a smooth $\frac{1}{2}$-density on Λ, we can consider its Lie derivative with respect to the vector field v_H restricted to Λ. We want an explicit formula for this Lie derivative in terms of local coordinates on X on a neighborhood over which Λ is horizontal.

Let

$$\iota : \Lambda \to T^*X$$

denote the embedding of Λ as submanifold of X so we are assuming that

$$\pi \circ \iota : \Lambda \to X$$

is a diffeomorphism. (We have replaced X by the appropriate neighborhood over which Λ is horizontal and on which we have coordinates $x^1, \ldots, x^m$.) We let $dx^{\frac{1}{2}}$ denote the standard $\frac{1}{2}$-density relative to these coordinates. Let a be a function on X, so that

$$\tau := (\pi \circ \iota)^* \left(a dx^{\frac{1}{2}}\right)$$

is a $\frac{1}{2}$-density on Λ, and the most general $\frac{1}{2}$-density on Λ can be written in this form. Our goal in this section is to compute the Lie derivative $D_{v_H}\tau$ and express it in a similar form. We will prove:

Proposition 3 *If* $\Lambda = \Lambda_\phi = \gamma_\phi(X)$ *then*

$$D_{v_H|\Lambda}(\pi \circ \iota)^* \left(a dx^{\frac{1}{2}}\right) = b(\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right)$$

where

$$b = D_{v_H|\Lambda}((\pi\circ\iota)^* a) + \iota^* \left[\frac{1}{2}\sum_{i,j} \frac{\partial^2 H}{\partial\xi_i \partial\xi_j}\frac{\partial^2\phi}{\partial x^i \partial x^j} + \frac{1}{2}\sum_i \frac{\partial^2 H}{\partial \xi_i \partial x^i}\right]((\pi\circ\iota)^* a). \tag{1.15}$$

Proof. Since $D_v(f\tau) = (D_v f)\tau + f D_v \tau$ for any vector field v, function f and any $\frac{1}{2}$-density τ, it suffices to prove (1.15) for the case the $a \equiv 1$ in which case the first term disappears. By Leibnitz's rule,

$$D_{v_H}(\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right) = \frac{1}{2} c (\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right)$$

where

$$D_{v_H}(\pi \circ \iota)^* |dx| = c(\pi \circ \iota)^* |dx|.$$

Here we are computing the Lie derivative of the density $(\pi \circ \iota)^* |dx|$, but we get the same function c if we compute the Lie derivative of the m-form

$$D_{v_H}(\pi\circ\iota)^*(dx^1 \wedge \cdots \wedge dx^m) = c(\pi\circ\iota)^*(dx^1\wedge\cdots\wedge dx^m).$$

Now $\pi^*(dx^1 \wedge \cdots \wedge dx^m)$ is a well defined m-form on T^*X and

$$D_{v_H|\Lambda}(\pi\circ\iota)^*(dx^1\wedge\cdots\wedge dx^m) = \iota^* D_{v_H}\pi^*(dx^1\wedge\cdots\wedge dx^m).$$

We may write dx^j instead of $\pi^* dx^j$ with no risk of confusion and we get

$$\begin{aligned} D_{v_H}(dx^1 \wedge \cdots \wedge dx^m) &= \sum_j dx^1 \wedge \cdots \wedge d(i(v_H)dx^j) \wedge \cdots \wedge dx^m \\ &= \sum_j dx^1 \wedge \cdots \wedge d\frac{\partial H}{\partial \xi_j} \wedge \cdots \wedge dx^m \\ &= \sum_j \frac{\partial^2 H}{\partial \xi_j \partial x^j} dx^1 \wedge \cdots \wedge dx^m + \\ &\quad \sum_{jk} dx^1 \wedge \cdots \wedge \frac{\partial^2 H}{\partial \xi_j \partial \xi_k} d\xi_k \wedge \cdots \wedge dx^m. \end{aligned}$$

We must apply ι^* which means that we must substitute $d\xi_k = d\left(\frac{\partial\phi}{\partial x^k}\right)$ into the last expression. We get

$$c = \sum_{i,j} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}\frac{\partial^2 \phi}{\partial x^i \partial x^j} + \sum_i \frac{\partial^2 H}{\partial \xi_i \partial x^i}$$

proving (1.15). □

1.3.2 The total symbol, locally.

Let U be an open subset of $\mathbb{R}^m$ and $x_1, \ldots x_m$ the standard coordinates. We will let D_j denote the differential operator

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}.$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ where the α_j are non-negative integers, we let

$$D^\alpha := D_1^{\alpha_1} \cdots D_m^{\alpha_m}$$

and

$$|\alpha| := \alpha_1 + \cdots + \alpha_m.$$

So the most general k-th order linear differential operator P can be written as

$$P = P(x, D) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha.$$

The **total symbol** of P is defined as

$$e^{-i\frac{x\cdot\xi}{\hbar}} P e^{i\frac{x\cdot\xi}{\hbar}} = \sum_{j=0}^{k} \hbar^{-j} p_j(x, \xi)$$

so that

$$p_j(x, \xi) = \sum_{|\alpha| = j} a_\alpha(x) \xi^\alpha. \tag{1.16}$$

So p_k is exactly the principal symbol as defined in (1.4).

Since we will be dealing with operators of varying orders, we will denote the principal symbol of P by

$$\sigma(P).$$

We should emphasize that the definition of the total symbol is heavily coordinate dependent: If we make a non-linear change of coordinates, the expression for the total symbol in the new coordinates will not look like the expression in the old coordinates. However the principal symbol *does* have an invariant expression as a function on the cotangent bundle which is a polynomial in the fiber variables.

1.3.3 The transpose of P.

We continue our study of linear differential operators on an open subset $U \subset \mathbb{R}^n$. If f and g are two smooth functions of compact support on U then

$$\int_U (Pf)g dx = \int_U f P^t g dx$$

where, by integration by parts,

$$P^t g = \sum (-1)^{|\alpha|} D^\alpha (a_\alpha g).$$

(Notice that in this definition, following convention, we are using g and not $\overline{g}$ in the definition of P^t.) Now

$$D^\alpha (a_\alpha g) = a_\alpha D^\alpha g + \cdots$$

where the $\cdots$ denote terms with fewer differentiations in g. In particular, the principal symbol of P^t is

$$p_k^t(x,\xi) = (-1)^k p_k(x,\xi). \tag{1.17}$$

Hence the operator

$$Q := \frac{1}{2}(P - (-1)^k P^t) \tag{1.18}$$

is of order $k-1$ The **sub-principal symbol** is defined as the principal symbol of Q (considered as an operator of degree $(k-1)$). So

$$\sigma_{sub}(P) := \sigma(Q)$$

where Q is given by (1.18).

1.3.4 The formula for the sub-principal symbol.

We claim that

$$\sigma_{sub}(P)(x,\xi) = p_{k-1}(x,\xi) + \frac{\sqrt{-1}}{2} \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i} p_k(x,\xi). \tag{1.19}$$

Proof. If $p_k(x,\xi) \equiv 0$, i.e. if P is actually an operator of degree $k-1$, then it follows from (1.17) (applied to $k-1$) and (1.18) that the principal symbol of Q is p_{k-1} which is the first term on the right in (1.19). So it suffices to prove (1.19) for operators which are strictly of order k. By linearity, it suffices to prove (1.19) for operators of the form

$$a_\alpha(x)D^\alpha.$$

By polarization it suffices to prove (1.19) for operators of the form

$$a(x)D^k, \quad D = \sum_{j=1}^{k} c_j D_j, \qquad c_i \in \mathbb{R}$$

and then, by making a linear change of coordinates, for an operator of the form

$$a(x)D_1^k.$$

For this operator

$$p_k(x,\xi) = a(x)\xi_1^k.$$

By Leibnitz's rule,

$$\begin{aligned}
P^t f &= (-1)^k D_1^k(af) \\
&= (-1)^k \sum_j \binom{k}{j} D_1^j a D_1^{k-j} f \\
&= (-1)^k \left(aD_1^k f + \frac{k}{i}\left(\frac{\partial a}{\partial x_1}\right) D^{k-1} f + \cdots \right) \quad \text{so} \\
Q &= \frac{1}{2}(P - (-1)^k P^t) \\
&= -\frac{k}{2i}\left(\frac{\partial a}{\partial x_1} D^{k-1} + \cdots\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\sigma(Q) &= \frac{ik}{2}\frac{\partial a}{\partial x_1}\xi_1^{k-1} \\
&= \frac{i}{2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_1}(a\xi_1^k) \\
&= \frac{i}{2}\sum_i \frac{\partial^2 p_k}{\partial x_i \partial \xi_i}(x,\xi). \qquad \square
\end{aligned}$$

1.3.5 The local expression for the transport operator R.

We claim that

$$\hbar^k e^{-i\phi/\hbar} P(ue^{i\phi/\hbar}) = p_k(x, d\phi)u + \hbar R u + \cdots$$

where R is the first order differential operator

$$Ru = \sum_j \frac{\partial p_k}{\partial \xi_j}(x,d\phi)D_j u + \left[\frac{1}{2\sqrt{-1}}\sum_{ij}\frac{\partial^2 p_k}{\partial\xi_i\partial\xi_j}(x,d\phi)\frac{\partial^2\phi}{\partial x_i\partial x_j}u + p_{k-1}(x,d\phi)\right]u. \tag{1.20}$$

Proof. The term coming from p_{k-1} is clearly the result of applying

$$\sum_{|\alpha|=k-1} a_\alpha D^\alpha.$$

So we only need to deal with a homogeneous operator of order k. Since the coefficients a_α are not going to make any difference in this formula, we need only prove it for the differential operator

$$P(x,D) = D^\alpha$$

which we will do by induction on $|\alpha|$.

For $|\alpha| = 1$ we have an operator of the form D_j and Leibnitz's rule gives

$$\hbar e^{-i\phi/\hbar}D_j(ue^{i\phi/\hbar}) = \frac{\partial\phi}{\partial x_j}u + \hbar D_j u$$

which is exactly (1.20) as $p_1(\xi) = \xi_j$, and so the second and third terms in (1.20) do not occur.

Suppose we have verified (1.20) for D^α and we want to check it for

$$D_r D^\alpha = D^{\alpha+\delta_r}.$$

So

$$\hbar^{|\alpha|+1}e^{-i\phi/\hbar}\left(D_r D^\alpha(ue^{i\phi/\hbar})\right) = \hbar e^{-i\phi/\hbar}D_r[(d\phi)^\alpha u e^{i\phi/\hbar} + \hbar(R_\alpha u)e^{i\phi/\hbar}] + \cdots$$

where R_α denotes the operator in (1.20) corresponding to D^α. A term involving the zero'th power of $\hbar$ can only come from applying the D_r to the exponential in the first expression and this will yield

$$(d\phi)^{\alpha+\delta_r}u$$

which $p_{|\alpha|+1}(d\phi)u$ as desired. In applying D_r to the second term in the square brackets and multiplying by $\hbar e^{-i\phi/\hbar}$ we get

$$\hbar^2 D_r(R_\alpha u) + \hbar\frac{\partial\phi}{\partial x_r}R_\alpha u$$

and we ignore the first term as we are ignoring all powers of $\hbar$ higher than the first. So all have to do is collect coefficients:

We have

$$D_r((d\phi^\alpha)u) = (d\phi)^\alpha D_r u + \frac{1}{\sqrt{-1}}\left[\alpha_1 (d\phi)^{\alpha-\delta_1}\frac{\partial^2\phi}{\partial x_1 \partial x_r} + \cdots + \alpha_m (d\phi)^{\alpha-\delta_m}\frac{\partial^2\phi}{\partial x_m \partial x_r}\right] u.$$

Also

$$\frac{\partial\phi}{\partial x_r} R_\alpha u =$$

$$\sum \alpha_i (d\phi)^{\alpha-\delta_i+\delta_r} D_i u + \frac{1}{2\sqrt{-1}}\sum_{ij}\alpha_i(\alpha_j - \delta_{ij})(d\phi)^{\alpha-\delta_i-\delta_j+\delta_r}\frac{\partial^2\phi}{\partial x_i \partial x_j} u.$$

The coefficient of $D_j u, \quad j \neq r$ is

$$\alpha_j (d\phi)^{(\alpha+\delta_r-\delta_j)}$$

as desired. The coefficient of $D_r u$ is

$$(d\phi)^\alpha + \alpha_r (d\phi)^\alpha = (\alpha_r + 1)(d\phi)^{(\alpha+\delta_r)-\delta_r}$$

as desired.

Let us now check the coefficient of $\frac{\partial^2\phi}{\partial x_i \partial x_j}$. If $i \neq r$ and $j \neq r$ then the desired result is immediate.

If $j = r$, there are two sub-cases to consider: 1) $j = r, j \neq i$ and 2) $i = j = r$.

If $j = r,\ j \neq i$ remember that the sum in R_α is over *all* i and j, so the coefficient of $\frac{\partial^2\phi}{\partial x_i \partial x_j}$ in

$$\sqrt{-1}\frac{\partial\phi}{\partial x_r} R_\alpha u$$

is

$$\frac{1}{2}\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)(d\phi)^{\alpha-\delta_i} = \alpha_i\alpha_j (d\phi)^{\alpha-\delta_i}$$

to which we add

$$\alpha_i (d\phi)^{\alpha-\delta_i}$$

to get

$$\alpha_i(\alpha_j+1)(d\phi)^{\alpha-\delta_i} = (\alpha+\delta_r)_i(\alpha+\delta_r)_j(d\phi)^{\alpha-\delta_i}$$

as desired.

If $i = j = r$ then the coefficient of $\frac{\partial^2\phi}{(\partial x_i)^2}$ in

$$\sqrt{-1}\frac{\partial\phi}{\partial x_r}R_\alpha u$$

is

$$\frac{1}{2}\alpha_i(\alpha_i - 1)(d\phi)^{\alpha-\delta_i}$$

to which we add

$$\alpha_i(d\phi)^{\alpha-\delta_i}$$

giving

$$\frac{1}{2}\alpha_i(\alpha_i + 1)(d\phi)^{\alpha-\delta_i}$$

as desired.

This completes the proof of (1.20).

1.3.6 Putting it together locally.

We have the following three formulas, some of them rewritten with H instead of p_k so as to conform with our earlier notation: The formula for the transport operator R given by (1.20):

$$\sum_j \frac{\partial H}{\partial \xi_j}(x, d\phi)D_j a + \left[\frac{1}{2\sqrt{-1}}\sum_{ij}\frac{\partial^2 H}{\partial\xi_i\partial\xi_j}(x, d\phi)\frac{\partial^2\phi}{\partial x_i\partial x_j} + p_{k-1}(x, d\phi)\right] a,$$

and the formula for the Lie derivative with respect to v_H of the pull back $(\pi\circ\iota)^*(adx^{\frac{1}{2}})$ given by $(\pi\circ\iota)*bdx^{\frac{1}{2}}$ where b is

$$\sum_j \frac{\partial H}{\partial\xi_j}(x, d\phi)\frac{\partial a}{\partial x_j} + \left[\frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial\xi_i\partial\xi_j}(x, d\phi)\frac{\partial^2\phi}{\partial x^i\partial x^j} + \frac{1}{2}\sum_i \frac{\partial^2 H}{\partial\xi_i\partial x^i}\right] a.$$

This is equation (1.15). Our third formula is the formula for the sub-principal symbol, equation (1.19), which says that

$$\sigma_{sub}(P)(x, \xi)a = \left[p_{k-1}(x, \xi) + \frac{\sqrt{-1}}{2}\sum_i \frac{\partial^2 H}{\partial x_i\partial\xi_i}(x, \xi)\right] a.$$

As first order partial differential operators on a, if we multiply the first expression above by $\sqrt{-1}$ we get the second plus $\sqrt{-1}$ times the third! So we can write the transport operator as

$$(\pi\circ\iota)^*[(Ra)dx^{\frac{1}{2}}] = \frac{1}{i}\left[D_{v_H} + i\sigma_{sub}(P)(x,d\phi)\right](\pi\circ\iota)^*(adx^{\frac{1}{2}}). \tag{1.21}$$

The operator inside the brackets on the right hand side of this equation is a perfectly good differential operator on $\frac{1}{2}$-densities on Λ. We thus have two questions to answer: Does this differential operator have invariant significance when Λ is horizontal - but in terms of a general coordinate transformation? Since the first term in the brackets comes from H and the symplectic form on the cotangent bundle, our question is one of attaching some invariant significance to the sub-principal symbol. We will deal briefly with this question in the next section and at more length in Chapter 6.

The second question is how to deal with the whole method - the eikonal equation, the transport equations, the meaning of the series in $\hbar$ etc. when we pass through a caustic. The answer to this question will occupy us for the whole book.

1.3.7 Differential operators on manifolds.

Differential operators on functions.

Let X be an m-dimensional manifold. An operator

$$P: \ C^\infty(X) \to C^\infty(X)$$

is called a differential operator of order k if, for every coordinate patch $(U, x_1, \ldots, x_m)$ the restriction of P to $C_0^\infty(U)$ is of the form

$$P = \sum_{|\alpha|\leq k} a_\alpha D^\alpha, \qquad a_\alpha \in C^\infty(U).$$

As mentioned above, the total symbol of P is no longer well defined, but the principal symbol *is* well defined as a function on T^*X. Indeed, it is defined as in Section 1.2.1: The value of the principal symbol H at a point $(x, d\phi(x))$ is determined by

$$H(x,d\phi(x))u(x) = \hbar^k e^{-i\frac{\phi}{\hbar}}(P(ue^{i\frac{\phi}{\hbar}})(x) + O(\hbar).$$

What about the transpose and the sub-principal symbol?

Differential operators on sections of vector bundles.

Let $E \to X$ and $F \to X$ be vector bundles. Let E be of dimension p and F be of dimension q. We can find open covers of X by coordinate patches $(U, x_1, \ldots, x_m)$ over which E and F are trivial. So we can find smooth sections $r_1, \ldots, r_p$ of E such that every smooth section of E over U can be written as

$$f_1 r_1 + \cdots f_p r_p$$

where the f_i are smooth functions on U and smooth sections $s_1, \ldots, s_q$ of F such that every smooth section of F over U can be written as

$$g_1 s_1 + \cdots + g_q s_q$$

over U where the g_j are smooth functions. An operator

$$P : \; C^\infty(X, E) \to C^\infty(X, F)$$

is called a differential operator of order k if, for every such U the restriction of P to smooth sections of compact support supported in U is given by

$$P\left(f_1 r_1 + \cdots f_p r_p\right) = \sum_{j=1}^{q} \sum_{i=1}^{p} P_{ij} f_i s_j$$

where the P_{ij} are differential operators of order k.

In particular if E and F are line bundles so that $p = q = 1$ it makes sense to talk of differential operators of order k from smooth sections of E to smooth sections of F. In a local coordinate system with trivializations r of E and s of F a differential operator locally is given by

$$fr \mapsto (Pf)s.$$

If $E = F$ and $r = s$ it is easy to check that the principal symbol of P is independent of the trivialization. (More generally the matrix of principal symbols in the vector bundle case is well defined up to appropriate pre and post multiplication by change of bases matrices, i.e. is well defined as a section of $\mathrm{Hom}(E, F)$ pulled up to the cotangent bundle. See Chaper II of [**?**] for the general discussion.)

In particular it makes sense to talk about a differential operator of degree k on the space of smooth $\frac{1}{2}$-densities and the principal symbol of such an operator.

The transpose and sub-principal symbol of a differential operator on $\frac{1}{2}$-densities.

If μ and ν are $\frac{1}{2}$-densities on a manifold X, their product $\mu \cdot \nu$ is a density (of order one). If this product has compact support, for example if μ or ν has compact support, then the integral

$$\int_X \mu \cdot \nu$$

is well defined. See Chapter 6 for details. So if P is a differential operator of degree k on $\frac{1}{2}$-densities, its transpose P^t is defined via

$$\int_X (P\mu) \cdot \nu = \int_X \mu \cdot (P^t \nu)$$

for all μ and ν one of which has compact support. Locally, in terms of a coordinate neighborhood $(U, x_1, \ldots, x_m)$, every $\frac{1}{2}$-density can be written as $f dx^{\frac{1}{2}}$ and then the local expression for P^t is given as in Section 1.3.3. We then define the operator Q as in equation (1.18) and the sub-principal symbol as the principal symbol of Q as an operator of degree $k-1$ just as in Section 1.3.3.

We have now answered our first question - that of giving a coordinate-free interpretation to the transport equation: Equation (1.21) makes good invariant sense if we agree that our differential operator is acting on $\frac{1}{2}$-densities rather than functions.

1.4 The plan.

We need to set up some language and prove various facts before we can return to our program of extending our method - the eikonal equation and the transport equations - so that they work past caustics.

In Chapter 2 we develop some necessary facts from symplectic geometry. In Chapter 3 we review some of the language of category theory. We also present a "baby" version of what we want to do later. We establish some facts about the category of finite sets and relations which will motivate similar constructions when we get to the symplectic "category" and its enhancement. We describe this symplectic "category" in Chapter 4. The objects in this "category" are symplectic manifolds and the morphisms are canonical relations. The quotation marks around the word "category" indicates that not all morphisms are composible.

In Chapter 5 we use this categorical language to explain how to find a local description of a Lagrangian submanifold of the cotangent bundle via "generating functions", a description which is valid even at caustics. The basic idea here goes back to Hamilton. But since this description depends on a choice, we must explain how to pass from one generating function to another. The main result here is the Hormander-Morse lemma which tells us that passage from one generating function to another can be accomplished by a series of "moves". The key analytic tool for proving this lemma is the method of stationary phase which we explain in Chapter 10. In Chapter 6 we study the calculus of $\frac{1}{2}$-densities, and in Chapter 7 we use half-densities to enhance the symplectic "category". In Chapter 8 we get to the main objects of study, which are oscillatory $\frac{1}{2}$-densities and develop their symbol calculus. In Chapter 9 we review the basic facts about the calculus of differential forms. In particular we review the Weil formula for the Lie derivative and the Moser trick for proving equivalence.

Chapter 2

Symplectic geometry.

2.1 Symplectic vector spaces.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbb{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbb{R}^2$ with

$$\omega_{\mathbb{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbb{R}^2$.

So if $u, v \in \mathbb{R}^2$ then $\omega_{\mathbb{R}^2}(u, v)$ is the oriented area of the parallelogram spanned by u and v.

2.1.1 Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V |\ \omega(v, w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Since $(W^\perp)^\perp = W$ by the non-degeneracy of ω, it follows that W is symplectic if and only if $W^\perp$ is. Also, the restriction of ω to any symplectic subspace W is non-degenerate, making W into a symplectic vector space. Conversely, to say that the restriction of ω to W is non-degenerate means precisely that $W \cap W^\perp = \{0\}$.

2.1.2 Normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e,f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbb{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence, by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots W_d$$

where $\dim V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbb{R}^2$ with its standard symplectic structure.

2.1.3 Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and if M is a complementary Lagrangian subspace, then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of V of the above form.

2.1.4 Consistent Hermitian structures.

In terms of the basis $e_1, \ldots, e_n, f_1, \ldots, f_d$ introduced above, consider the linear map

$$J : \quad e_i \mapsto -f_i, \quad f_i \mapsto e_i.$$

It satisfies

$$\begin{aligned} J^2 &= -I, && (2.1)\\ \omega(Ju, Jv) &= \omega(u, v), \quad \text{and} && (2.2)\\ \omega(Ju, v) &= \omega(Jv, u). && (2.3) \end{aligned}$$

Notice that any J which satisfies two of the three conditions above automatically satisfies the third. Condition (2.1) says that J makes V into a d-dimensional complex vector space. Condition (2.2) says that J is a symplectic transformation, i.e acts so as to preserve the symplectic form ω. Condition (2.3) says that $\omega(Ju, v)$ is a real symmetric bilinear form.

All three conditions (really any two out of the three) say that $(\ ,\ ) = (\ ,\ )_{\omega,J}$ defined by

$$(u, v) = \omega(Ju, v) + i\omega(u, v)$$

is a semi-Hermitian form whose imaginary part is ω. For the J chosen above this form is actually Hermitian, that is the real part of $(\ ,\ )$ is positive definite.

2.2 Lagrangian complements.

The results of this section will be used extensively, especially in Chapter 5.

Let V be a symplectic vector space.

Proposition 4 *Given any finite collection of Lagrangian subspaces $M_1, \dots, M_k$ of V one can find a Lagrangian subspace L such that*

$$L \cap M_j = \{0\}, \quad i = 1, \dots k.$$

Proof. We can always find an isotropic subspace L with $L \cap M_j = \{0\}, \quad i = 1, \dots k$, for example a line which does not belong to any of these subspaces. Suppose that L is an isotropic subspace with $L \cap M_j = \{0\}$, $\forall j$ and is not properly contained in a larger isotropic subspace with this property. We claim that L is Lagrangian. Indeed, if not, $L^\perp$ is a coisotropic subspace which strictly contains L. Let $\pi : L^\perp \to L^\perp/L$ be the quotient map. Each of the spaces $\pi(L^\perp \cap M_j)$ is an isotropic subspace of the symplectic vector space $L^\perp/L$ and so each of these spaces has positive codimension. So we can choose a line ℓ in $L^\perp/L$ which does not intersect any of the $\pi(L^\perp \cap M_j)$. Then $L' := \pi^{-1}(\ell)$ is an isotropic subspace of $L^\perp \subset V$ with $L \cap M_j = \{0\}$, $\forall\, j$ and strictly containing L, a contradiction. □

In words, given a finite collection of Lagrangian subspaces, we can find a Lagrangian subspace which is transversal to all of them.

2.2.1 Choosing Lagrangian complements "consistently".

The results of this section are purely within the framework of symplectic linear algebra. Hence their logical place is here. However their main interest is that they serve as lemmas for more geometrical theorems, for example the Weinstein isotropic embedding theorem. The results here all have to do with making choices in a "consistent" way, so as to guarantee, for example, that the choices can be made to be invariant under the action of a group.

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2}\dim V$$

and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2}\dim V \quad \text{and } W \cap L = \{0\}$$

can be written as

$$\text{graph}(A)$$

where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

Now

$$\text{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u, w) - C(w, u) = -\omega(u, w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 5 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Here the word "consistent" means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and the positive definite symmetric form B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\, u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = B(u, v) = B(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

and

$$\omega(Ju, JJv) = -\omega(Ju, v) = -\omega(Jv, u) = \omega(Jv, JJu)$$

so (2.3) holds for all $u, v \in V$. We should write $J_{B,L}$ for this complex structure, or J_L when B is understood

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L \tag{2.4}$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}. Since B is invariant under T the right hand side equals $B(u, v) = \omega(u, J_L v) = \omega(Tu, TJ_L v)$ since ω is invariant under T. Thus

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_L v)$$

showing that

$$TJ_L = J_{TL}T$$

when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form $J_L u$ where $u \in L$ and $TJ_L u \in TL'$ so

$$J_{TL}TJ_L u = -J_{TL}^{-1}TJ_L u = -Tu = TJ_L(J_L u). \quad \square$$

Let I be an isotropic subspace of V and let $I^\perp$ be its symplectic orthogonal subspace so that $I \subset I^\perp$. Let

$$I_B = (I^\perp)^B$$

be the B-orthogonal complement to $I^\perp$. Thus

$$\dim I_B = \dim I$$

and since $I_B \cap I^\perp = \{0\}$, the spaces I_B and I are non-singularly paired under ω. In other words, the restriction of ω to $I_B \oplus I$ is symplectic. The proof of the preceding proposition gives a Lagrangian complement (inside $I_B \oplus I$) to I which, as a subspace of V has zero intersection with $I^\perp$. We have thus proved:

Proposition 6 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning an isotropic complement I' to every co-isotropic subspace $I^\perp$.*

We can use the preceding proposition to prove the following:

Proposition 7 *Let V_1 and V_2 be symplectic vector spaces of the same dimension, with $I_1 \subset V_1$ and $I_2 \subset V_2$ isotropic subspaces, also of the same dimension. Suppose we are given*

- *a linear isomorphism* $\lambda : I_1 \to I_2$ *and*
- *a symplectic isomorphism* $\ell : I_1^\perp / I_1 \to I_2^\perp / I_2$.

Then there is a symplectic isomorphism

$$\gamma : V_1 \to V_2$$

such that

1. $\gamma : I_1^\perp \to I_2^\perp$ *and (hence)* $\gamma : I_1 \to I_2$,
2. *The map induced by* γ *on* $I_1^\perp / I_1$ *is* ℓ *and*
3. *The restriction of* γ *to* I_1 *is* λ.

Furthermore, in the presence of positive definite symmetric bilinear forms B_1 *on* V_1 *and* B_2 *on* V_2 *the choice of* γ *can be made in a "canonical" fashion.*

Indeed, choose isotropic complements I_{1B} to $I_1^\perp$ and I_{2B} to $I_2^\perp$ as given by the preceding proposition, and also choose B orthogonal complements Y_1 to I_1 inside $I_1^\perp$ and Y_2 to I_2 inside $I_2^\perp$. Then Y_i $(i = 1, 2)$ is a symplectic subspace of V_i which can be identified as a symplectic vector space with $I_i^\perp / I_i$. We thus have

$$V_1 = (I_1 \oplus I_{1B}) \oplus Y_1$$

as a direct sum decomposition into the sum of the two symplectic subspaces $(I_1 \oplus I_{1B})$ and Y_1 with a similar decomposition for V_2. Thus ℓ gives a symplectic isomorphism of $Y_1 \to Y_2$. Also

$$\lambda \oplus (\lambda^*)^{-1} : I_1 \oplus I_{1B} \to I_2 \oplus I_{2B}$$

is a symplectic isomorphism which restricts to λ on I_1. $\square$

2.3 Equivariant symplectic vector spaces.

Let V be a symplectic vector space. We let $Sp(V)$ denote the group of all all symplectic automorphisms of V, i.e all maps T which satisfy $\omega(Tu, Tv) = \omega(u, v)\ \forall\, u, v \in V$.

A representation $\tau : G \to \mathrm{Aut}(V)$ of a group G is called symplectic if in fact $\tau : G \to Sp(V)$. Our first task will be to show that if G is compact, and τ is symplectic, then we can find a J satisfying (2.1) and (2.2), which commutes with all the $\tau(a)$, $a \in G$ and such that the associated Hermitian form is positive definite.

2.3.1 Invariant Hermitian structures.

Once again, let us start with a positive definite symmetric bilinear form B. By averaging over the group we may assume that B is G invariant. (Here is where we use the compactness of G.) Then there is a unique linear operator K such that

$$B(Ku, v) = \omega(u, v) \quad \forall\, u, v \in V.$$

Since both B and ω are G-invariant, we conclude that K commutes with all the $\tau(a)$, $a \in G$. Since $\omega(v, u) = -\omega(u, v)$ we conclude that K is skew adjoint relative to B, i.e. that

$$B(Ku, v) = -B(u, Kv).$$

Also K is non-singular. Then K^2 is symmetric and non-singular, and so V can be decomposed into a direct sum of eigenspaces of K^2 corresponding to distinct eigenvalues, all non-zero. These subspaces are mutually orthogonal under B and invariant under G. If $K^2u = \mu u$ then

$$\mu B(u, u) = B(K^2u, u) = -B(Ku, Ku) < 0$$

so all these eigenvalues are negative; we can write each μ as $\mu = -\lambda^2$, $\lambda > 0$. Furthermore, if $K^2u = -\lambda^2 u$ then

$$K^2(Ku) = KK^2u = -\lambda^2 Ku$$

so each of these eigenspaces is invariant under K. Also, any two subspaces corresponding to different values of λ^2 are orthogonal under ω. So we need only define J on each such subspace so as to commute with all the $\tau(a)$ and so as to satisfy (2.1) and (2.2), and then extend linearly. On each such subspace set

$$J := \lambda K^{-1}.$$

Then (on this subspace)

$$J^2 = \lambda^2 K^{-2} = -I$$

and

$$\omega(Ju, v) = \lambda\omega(K^{-1}u, v) = \lambda B(u, v)$$

is symmetric in u and v. Furthermore $\omega(Ju, u) = \lambda B(u, u) > 0$. □

Notice that if τ is irreducible, then the Hermitian form $(\ ,\ ) = \omega(J\cdot, \cdot) + i\omega(\cdot, \cdot)$ is uniquely determined by the property that its imaginary part is ω.

2.3.2 The space of fixed vectors for a compact group of symplectic automorphisms is symplectic.

If we choose J as above, if $\tau(a)u = u$ then $\tau(a)Ju = Ju$. So the space of fixed vectors is a complex subspace for the complex structure determined by J. But the restriction of a positive definite Hermitian form to any (complex) subspace is again positive definite, in particular non-singular. Hence its imaginary part, the symplectic form ω, is also non-singular. □
This result need not be true if the group is not compact. For example, the one parameter group of shear transformations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

in the plane is symplectic as all of these matrices have determinant one. But the space of fixed vectors is the x-axis.

2.3.3 Toral symplectic actions.

Suppose that $G = \mathbf{T}^n$ is an n-dimensional torus, and that $\mathbf{g}$ denotes its Lie algebra. Then $\exp:\mathbf{g} \to G$ is a surjective homomorphisms, whose kernel $\mathbb{Z}_G$ is a lattice.

If $\tau : G \to U(V)$ as above, we can decompose V into a direct sum of one dimensional complex subspaces

$$V = V_1 \oplus \cdots \oplus V_d$$

where the restriction of τ to each subspace is given by

$$\tau_{|V_k}(\exp\ \xi)v = e^{2\pi i\alpha_k(\xi)}v$$

where

$$\alpha_k \in \mathbb{Z}_G^*,$$

the dual lattice.

2.4 Symplectic manifolds.

Recall that a manifold M is called **symplectic** if it comes equipped with a closed non-degenerate two form ω. A diffeomorphism is called symplectic if it preserves ω. We shall usually shorten the phrase "symplectic diffeomorphism" to **symplectomorphism**

A vector field v is called symplectic if

$$D_v\omega = 0.$$

Since $D_v\omega = d\iota(v)\omega + \iota(v)d\omega = d\iota(v)\omega$ as $d\omega = 0$, a vector field v is symplectic if and only if $\iota(v)\omega$ is closed.

Recall that a vector field v is called **Hamiltonian** if $\iota(v)\omega$ is exact. If θ is a closed one form, and v a vector field, then $D_v\theta = d\iota(v)\theta$ is exact. Hence if v_1 and v_2 are symplectic vector fields

$$D_{v_1}\iota(v_2)\omega = \iota([v_1, v_2])\omega$$

so $[v_1, v_2]$ is Hamiltonian with

$$\iota([v_1, v_2])\omega = d\omega(v_2, v_1).$$

2.5 Darboux style theorems.

These are theorems which state that two symplectic structures on a manifold are the same or give a normal form near a submanifold etc. We will prove them using the Moser-Weinstein method. This method hinges on the basic formula of differential calculus: If $f_t : X \to Y$ is a smooth family of maps and ω_t is a one parameter family of differential forms on Y then

$$\frac{d}{dt} f_t^* \omega_t = f_t^* \frac{d}{dt}\omega_t + Q_t d\omega_t + dQ_t\omega_t \tag{2.5}$$

where

$$Q_t : \Omega^k(Y) \to \Omega^{k-1}(X)$$

is given by

$$Q_t\tau(w_1, \ldots, w_{k-1}) := \tau(v_t, df_t(w_1), \ldots, df_t(w_{k-1}))$$

where

$$v_t : X \to T(Y), \quad v_t(x) := \frac{d}{dt} f_t(x).$$

If ω_t does not depend explicitly on t then the first term on the right of (2.5) vanishes, and integrating (2.5) with respect to t from 0 to 1 gives

$$f_1^* - f_0^* = dQ + Qd, \quad Q := \int_0^1 Q_t dt. \tag{2.6}$$

We give a review of all of this in Chapter 9.

Here is the first Darboux type theorem:

2.5.1 Compact manifolds.

Theorem 1 *Let M be a compact manifold, ω_0 and ω_1 two symplectic forms on M in the same cohomology class so that*

$$\omega_1 - \omega_0 = d\alpha$$

for some one form α. Suppose in addition that

$$\omega_t := (1-t)\omega_0 + t\omega_1$$

is symplectic for all $0 \leq t \leq 1$. Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Solve the equation

$$\iota(v_t)\omega_t = -\alpha$$

which has a unique solution v_t since ω_t is symplectic. Then solve the time dependent differential equation

$$\frac{df_t}{dt} = v_t(f_t), \quad f_0 = \mathrm{id}$$

which is possible since M is compact. Since

$$\frac{d\omega_t}{dt} = d\alpha,$$

the fundamental formula (2.5) gives

$$\frac{df_t^*\omega_t}{dt} = f_t^*\left[d\alpha + 0 - d\alpha\right] = 0$$

so

$$f_t^*\omega_t \equiv \omega_0.$$

In particular, set $t = 1$. □

This style of argument was introduced by Moser and applied to Darboux type theorems by Weinstein.

Here is a modification of the above:

Theorem 2 *Let M be a compact manifold, and ω_t, $0 \leq t \leq 1$ a family of symplectic forms on M in the same cohomology class.*

Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Break the interval $[0, 1]$ into subintervals by choosing $t_0 = 0 < t_1 < t_2 < \cdots < t_N = 1$ and such that on each subinterval the "chord" $(1-s)\omega_{t_i} + s\omega_{t_{i+1}}$ is close enough to the curve $\omega_{(1-s)t_i+st_{i+1}}$ so that the forms $(1-s)\omega_{t_i}+s\omega_{t_{i+1}}$ are symplectic. Then successively apply the preceding theorem. □

2.5.2 Compact submanifolds.

The next version allows M to be non-compact but has to do with with behavior near a compact submanifold. We will want to use the following proposition:

Proposition 8 *Let X be a compact submanifold of a manifold M and let*

$$i : X \to M$$

denote the inclusion map. Let $\gamma \in \Omega^k(M)$ be a k-form on M which satisfies

$$\begin{aligned} d\gamma &= 0 \\ i^*\gamma &= 0. \end{aligned}$$

Then there exists a neighborhood U of X and a $k-1$ form β defined on U such that

$$\begin{aligned} d\beta &= \gamma \\ \beta_{|X} &= 0. \end{aligned}$$

(This last equation means that at every point $p \in X$ we have

$$\beta_p(w_1, \ldots, w_{k-1}) = 0$$

for all tangent vectors, not necessarily those tangent to X. So it is a much stronger condition than $i^*\beta = 0$.)

Proof. By choice of a Riemann metric and its exponential map, we may find a neighborhood of W

of X in M and a smooth retract of W onto X, that is a one parameter family of smooth maps

$$r_t : W \to W$$

and a smooth map $\pi : W \to X$ with

$$r_1 = \mathrm{id}, \quad r_0 = i \circ \pi, \ \pi : W \to X, \ \ r_t \circ i \equiv i.$$

Write

$$\frac{dr_t}{dt} = w_t \circ r_t$$

and notice that $w_t \equiv 0$ at all points of X. Hence the form

$$\beta := Q\gamma$$

has all the desired properties where Q is as in (2.6). □

Theorem 3 *Let X, M and i be as above, and let ω_0 and ω_1 be symplectic forms on M such that*

$$i^*\omega_1 = i^*\omega_0$$

and such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$. Then there exists a neighborhood U of M and a smooth map

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{ and } \quad f^*\omega_0 = \omega_1.$$

Proof. Use the proposition to find a neighborhood W of X and a one form α defined on W and vanishing on X such that

$$\omega_1 - \omega_0 = d\alpha$$

on W. Let v_t be the solution of

$$\iota(v_t)\omega_t = -\alpha$$

where $\omega_t = (1-t)\omega_0 + t\omega_1$. Since v_t vanishes identically on X, we can find a smaller neighborhood of X if necessary on which we can integrate v_t for $0 \leq t \leq 1$ and then apply the Moser argument as above. □

A variant of the above is to assume that we have a curve of symplectic forms ω_t with $i^*\omega_t$ independent of t.

Finally, a very useful variant is Weinstein's

Theorem 4 *X, M, i as above, and ω_0 and ω_1 two symplectic forms on M such that $\omega_{1|X} = \omega_{0|X}$. Then there exists a neighborhood U of M and a smooth map*

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{and} \quad f^*\omega_0 = \omega_1.$$

Here we can find a neighborhood of X such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$ since X is compact. □

One application of the above is to take X to be a point. The theorem then asserts that all symplectic structures of the same dimension are locally symplectomorphic. This is the original theorem of Darboux.

2.5.3 The isotropic embedding theorem.

Another important application of the preceding theorem is Weinstein's isotropic embedding theorem: Let (M, ω) be a symplectic manifold, X a compact manifold, and $i : X \to M$ an isotropic embedding, which means that $di_x(TX)_x$ is an isotropic subspace of $TM_{i(x)}$ for all $x \in X$. Thus

$$di_x(TX)_x \subset (di_x(TX)_x)^\perp$$

where $(di_x(TX)_x)^\perp$ denotes the orthogonal complement of $di_x(TX)_x$ in $TM_{i(x)}$ relative to $\omega_{i(x)}$. Hence

$$(di_x(TX)_x)^\perp / di_x(TX)_x$$

is a symplectic vector space, and these fit together into a symplectic vector bundle (i.e. a vector bundle with a symplectic structure on each fiber). We will call this the symplectic normal bundle of the embedding, and denote it by

$$SN_i(X)$$

or simply by $SN(X)$ when i is taken for granted.

Suppose that U is a neighborhood of $i(X)$ and $g : U \to N$ is a symplectomorphism of U into a second symplectic manifold N. Then $j = g \circ i$ is an isotropic embedding of X into N and f induces an isomorphism

$$g_* : NS_i(X) \to NS_j(X)$$

of symplectic vector bundles. Weinstein's isotropic embedding theorem asserts conversely, any isomorphism between symplectic normal bundles is in fact induced by a symplectomorphism of a neighborhood of the image:

Theorem 5 *Let (M, ω_M, X, i) and (N, ω_N, X, j) be the data for isotropic embeddings of a compact manifold X. Suppose that*

$$\ell : SN_i(X) \to SN_j(X)$$

is an isomorphism of symplectic vector bundles. Then there is a neighborhood U of $i(X)$ in M and a symplectomorphism g of U onto a neighborhood of $j(X)$ in N such that

$$g_* = \ell.$$

For the proof, we will need the following extension lemma:

Proposition 9 *Let*

$$i : X \to M, \quad j : Y \to N$$

be embeddings of compact manifolds X and Y into manifolds M and N. suppose we are given the following data:

- *A smooth map $f : X \to Y$ and, for each $x \in X$,*
- *A linear map $A_x TM_{i(x)} \to TN_{j(f(x))}$ such that the restriction of A_x to $TX_x \subset TM_{i(x)}$ coincides with df_x.*

Then there exists a neighborhood W of X and a smooth map $g : W \to N$ such that

$$g \circ i = f \circ i$$

and

$$dg_x = A_x \quad \forall\ x \in X.$$

Proof. If we choose a Riemann metric on M, we may identify (via the exponential map) a neighborhood of $i(X)$ in M with a section of the zero section of X in its (ordinary) normal bundle. So we may assume that $M = \mathcal{N}_i X$ is this normal bundle. Also choose a Riemann metric on N, and let

$$\exp : \mathcal{N}_j(Y) \to N$$

be the exponential map of this normal bundle relative to this Riemann metric. For $x \in X$ and $v \in N_i(i(x))$ set

$$g(x,v) := \exp_{j(x)}(A_x v).$$

Then the restriction of g to X coincides with f, so that, in particular, the restriction of dg_x to the tangent space to T_x agrees with the restriction of A_x to this subspace, and also the restriction of dg_x to the normal space to the zero section at x agrees A_x so g fits the bill. □

Proof of the theorem. We are given linear maps $\ell_x : (I_x^\perp/I_x) \to J_x^\perp/J_x$ where $I_x = di_x(TX)_x$ is an isotropic subspace of $V_x := TM_{i(x)}$ with a similar notation involving j. We also have the identity map of

$$I_x = TX_x = J_x.$$

So we may apply Proposition 7 to conclude the existence, for each x of a unique symplectic linear map

$$A_x : \; TM_{i(x)} \to TN_{j(x)}$$

for each $x \in X$. We may then extend this to an actual diffeomorphism, call it h on a neighborhood of $i(X)$, and since the linear maps A_x are symplectic, the forms

$$h^*\omega_N \quad \text{and} \quad \omega_M$$

agree at all points of X. We then apply Theorem 4 to get a map k such that $k^*(h^*\omega_N) = \omega_M$ and then $g = h \circ k$ does the job. □

Notice that the constructions were all determined by the choice of a Riemann metric on M and of a Riemann metric on N. So if these metrics are invariant under a group G, the corresponding g will be a G-morphism. If G is compact, such invariant metrics can be constructed by averaging over the group.

An important special case of the isotropic embedding theorem is where the embedding is not merely isotropic, but is Lagrangian. Then the symplectic normal bundle is trivial, and the theorem asserts that all Lagrangian embeddings of a compact manifold are locally equivalent, for example equivalent to the embedding of the manifold as the zero section of its cotangent bundle.

Chapter 3

The language of category theory.

3.1 Categories.

We briefly recall the basic definitions:

A **category C** consists of the following data:

(i) A set, $Ob(\mathbf{C})$, whose elements are called the **objects** of **C** ,

(ii) For every pair (X, Y) of $Ob(\mathbf{C})$ a set, $\mathrm{Morph}(X, Y)$, whose elements are called the **morphisms** or **arrows** from X to Y,

(iii) For every triple (X, Y, Z) of $Ob(\mathbf{C})$ a map from $\mathrm{Morph}(X, Y) \times \mathrm{Morph}(Y, Z)$ to $\mathrm{Morph}(X, Z)$ called the **composition map** and denoted $(f, g) \rightsquigarrow g \circ f$.

These data are subject to the following conditions:

(iv) The composition of morphisms is *associative*

(v) For each $X \in Ob(\mathbf{C})$ there is an $id_X \in \mathrm{Morph}(X, X)$ such that

$$f \circ id_X = f, \ \forall f \in \mathrm{Morph}(X, Y)$$

(for any Y) and

$$id_X \circ f = f, \ \forall f \in \mathrm{Morph}(Y, X)$$

(for any Y).

It follows from the definitions that id_X is unique.

3.2 Functors and morphisms.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

(vi) a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

and

(vii) for each pair (X, Y) of $Ob(\mathcal{C})$ a map

$$F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$$

subject to the rules

(viii)

$$F(id_X) = id_{F(X)}$$

and

(ix)

$$F(g \circ f) = F(g) \circ F(f).$$

This is what is usually called a **covariant functor**.

A **contravariant functor** would have $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(Y), F(X))$ in (vii) and $F(f) \circ F(g)$ on the right hand side of (ix).)

Here is an important example, valid for any category $\mathcal{C}$. Let us fix an $X \in Ob(\mathcal{C})$. We get a functor

$$F_X : \mathcal{C} \to \mathbf{Set}$$

by the rule which assigns to each $Y \in Ob(\mathcal{C})$ the set $F_X(Y) = \mathrm{Hom}(X, Y)$ and to each $f \in \mathrm{Hom}(Y, Z)$ the map $F_X(f)$ consisting of composition (on the left) by f. In other words, $F_X(f) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ is given by

$$g \in \mathrm{Hom}(X, Y) \mapsto f \circ g \in \mathrm{Hom}(X, Z).$$

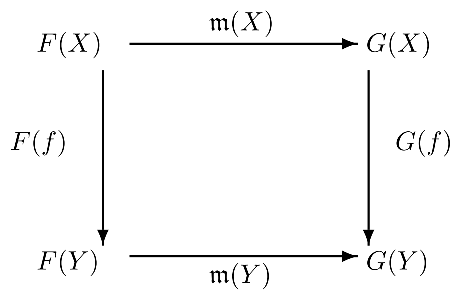


Figure 3.1:

Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A **morphism**, $\mathfrak{m}$, from F to G (older name: "natural transformation") consists of the following data:

(x) for each $X \in Ob(\mathcal{C})$ an element $\mathfrak{m}(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ subject to the "naturality condition"

(xi) for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the diagram in Figure 3.1 commutes. In other words

$$\mathfrak{m}(Y) \circ F(f) = G(f) \circ \mathfrak{m}(X) \quad \forall f \in f \in \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

3.2.1 Involutory functors and involutive functors.

Consider the category $\mathcal{V}$ whose objects are finite dimensional vector spaces (over some given field $\mathbb{K}$) and whose morphisms are linear transformations. We can consider the "transpose functor" $F : \mathcal{V} \to \mathcal{V}$ which assigns to every vector space V its dual space

$$V^* = \mathrm{Hom}(V, \mathbb{K})$$

and which assigns to every linear transformation $\ell : V \to W$ its transpose

$$\ell^* : V^* \to W^*.$$

In other words,

$$F(V) = V^*, \quad F(\ell) = \ell^*.$$

This is a contravariant functor which has the property that F^2 is naturally equivalent to the identity functor. There does not seem to be a standard name for this type of functor. We will call it an **involutory** functor.

A special type of involutory functor is one in which $F(X) = X$ for all objects X and $F^2 = \mathrm{id}$ (not merely naturally equivalent to the identity). We shall call such a functor a **involutive** functor. We will refer to a category with an involutive functor as an **involutive category**, or say that we have a category with an involutive structure.

For example, let $\mathcal{H}$ denote the category whose objects are Hilbert spaces and whose morphisms are bounded linear transformations. We take $F(X) = X$ on objects and $F(L) = L^\dagger$ on bounded linear transformations where $L^\dagger$ denotes the adjoint of L in the Hilbert space sense.

3.3 Example: Sets, maps and relations.

The category **Set** is the category whose objects are ("all") sets and and whose morphisms are ("all") maps between sets. For reasons of logic, the word "all" must be suitably restricted to avoid contradiction.

We will take the extreme step in this section of restricting our attention to the class of finite sets. Our main point is to examine a category whose objects are finite sets, but whose morphisms are much more general than maps. Some of the arguments and constructions that we use in the study of this example will be models for arguments we will use later on, in the context of the symplectic "category".

3.3.1 The category of finite relations.

We will consider the category whose objects are finite sets. But we enlarge the set of morphisms by defining

$$\mathrm{Morph}(X, Y) = \text{the collection of all subsets of } X \times Y.$$

A subset of $X \times Y$ is called a **relation**. We must describe the map

$$\mathrm{Morph}(X, Y) \times \mathrm{Morph}(Y, Z) \to \mathrm{Morph}(X, Z)$$

and show that this composition law satisfies the axioms of a category. So let

$$\Gamma_1 \in \mathrm{Morph}(X, Y) \quad \text{and } \Gamma_2 \in \mathrm{Morph}(Y, Z).$$

Define

$$\Gamma_2 \circ \Gamma_1 \subset X \times Z$$

by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \Leftrightarrow \exists\, y \in Y \text{ such that } (x, y) \in \Gamma_1 \text{ and } (y, z) \in \Gamma_2. \tag{3.1}$$

Notice that if $f : X \to Y$ and $g : Y \to Z$ are maps, then

$$\mathrm{graph}(f) = \{(x, f(x)\} \in \mathrm{Morph}(X, Y) \quad \text{and} \qquad \mathrm{graph}(g) \in \mathrm{Morph}(Y, Z)$$

with

$$\mathrm{graph}(g) \circ \mathrm{graph}(f) = \mathrm{graph}(g \circ f).$$

So we have indeed enlarged the category of finite sets and maps.

We still must check the axioms. Let $\Delta_X \subset X \times X$ denote the diagonal:

$$\Delta_X = \{(x, x),\ x \in X\},$$

so

$$\Delta_X \in \mathrm{Morph}(X, X).$$

If $\Gamma \in \mathrm{Morph}(X, Y)$ then

$$\Gamma \circ \Delta_X = \Gamma \quad \text{and } \Delta_Y \circ \Gamma = \Gamma.$$

So Δ_X satisfies the conditions for id_X.

Let us now check the associative law. Suppose that $\Gamma_1 \in \mathrm{Morph}(X, Y), \Gamma_2 \in \mathrm{Morph}(Y, Z)$ and $\Gamma_3 \in \mathrm{Morph}(Z, W)$. Then both $\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$ and $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1$ consist of all $(x, w) \in X \times W$ such that there exist $y \in Y$ and $z \in Z$ with

$$(x, y) \in \Gamma_1,\ (y, z) \in \Gamma_2, \quad \text{and } (z, w) \in \Gamma_3.$$

This proves the associative law.

Let us call this category **FinRel**.

3.3.2 Categorical "points".

Let us pick a distinguished one element set and call it "pt.". Giving a *map* from pt. to any set X is the same as picking a point of X. So in the category **Set** of sets and *maps*, the points of X are the same as the morphisms from our distinguished object pt. to X.

In a more general category, where the objects are not necessarily sets, we can not talk about the points of an object X. However if we have a distinguished object pt., then we can *define* a **"point"** of any object X to be an element of $\mathrm{Morph}(\mathrm{pt.}, X)$. For example, later on, when we study the symplectic "category" whose objects are symplectic manifolds, we will find that the "points" in a symplectic manifold are its Lagrangian submanifolds. This idea has been emphasized by Weinstein. As he points out, this can be considered as a manifestation of the Heisenberg uncertainty principle in symplectic geometry.

In the category **FinRel**, the category of finite sets and relations, an element of $\mathrm{Morph}(\mathrm{pt.}, X)$, i.e a subset of pt. $\times X$ is the same as a subset of X (by projection onto the second factor). So in this category, the "points" of X are the subsets of X. Many of the constructions we do here can be considered as warm ups to similar constructions in the symplectic "category".

A morphism $\Gamma \in \mathrm{Morph}(X, Y)$ yields a map from "points" of X to "points" of Y.

Consider three objects X, Y, Z. Inside

$$X \times X \times Y \times Y \times Z \times Z$$

we have the subset

$$\Delta_X \times \Delta_Y \times \Delta_Z.$$

Let us move the first X factor past the others until it lies to immediate left of the right Z factor, so consider the subset

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$

By introducing parentheses around the first four and last two factors we can write

$$\tilde{\Delta}_{X,Y,Z} \subset (X \times Y \times Y \times Z) \times (X \times Z).$$

In other words,

$$\tilde{\Delta}_{X,Y,Z} \in \mathrm{Morph}(X \times Y \times Y \times Z, X \times Z).$$

Let $\Gamma_1 \in \text{Morph}(X, Y)$ and $\Gamma_2 \in \text{Morph}(Y, Z)$. Then

$$\Gamma_1 \times \Gamma_2 \subset X \times Y \times Y \times Z$$

is a "point" of $X \times Y \times Y \times Z$. We identify this "point" with an element of

$$\text{Morph}(\text{pt.}, X \times Y \times Y \times Z)$$

so that we can form

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2)$$

which consists of all (x, z) such that

$$\exists (x_1, y_1, y_2, z_1, x, z) \text{ with}$$

$$\begin{aligned} (x_1, y_1) &\in \Gamma_1, \\ (y_2, z_1) &\in \Gamma_2, \\ x_1 &= x, \\ y_1 &= y_2, \\ z_1 &= z. \end{aligned}$$

Thus

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1. \tag{3.2}$$

Similarly, given four sets X, Y, Z, W we can form

$$\tilde{\Delta}_{X,Y,Z,W} \subset (X \times Y \times Y \times Z \times Z \times W) \times (X \times W)$$

$$\tilde{\Delta}_{X,Y,Z,W} = \{(x, y, y, z, z, w, x, w)\}$$

so

$$\tilde{\Delta}_{X,Y,Z,W} \in \text{Morph}(X \times Y \times Y \times Z \times Z \times W, X \times W).$$

If $\Gamma_1 \in \text{Morph}(X, Y)$, $\Gamma_2 \in \text{Morph}(Y, Z)$, and $\Gamma_3 \in \text{Morph}(Z, W)$ then

$$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = (\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \tilde{\Delta}_{X,Y,Z,W}(\Gamma_1 \times \Gamma_2 \times \Gamma_3).$$

From this point of view the associative law is a reflection of the fact that

$$(\Gamma_1 \times \Gamma_2) \times \Gamma_3 = \Gamma_1 \times (\Gamma_2 \times \Gamma_3) = \Gamma_1 \times \Gamma_2 \times \Gamma_3.$$

3.3.3 The transpose.

In our category **FinRel**, if $\Gamma \in \mathrm{Morph}(X,Y)$ define $\Gamma^\dagger \in \mathrm{Morph}(Y,X)$ by

$$\Gamma^\dagger := \{(y,x)|(x,y) \in \Gamma\}.$$

We have defined a map

$$\dagger : \mathrm{Morph}(X,Y) \to \mathrm{Morph}(Y,X) \tag{3.3}$$

for all objects X and Y which clearly satisfies

$$\dagger^2 = \mathrm{id} \tag{3.4}$$

and

$$(\Gamma_2 \circ \Gamma_1)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger. \tag{3.5}$$

So $\dagger$ is a contravariant functor and satisfies our conditions for an involution. This makes our category **FinRel** of finite sets and relations into an involutive category.

3.3.4 The finite Radon transform.

This is a contravariant functor $\mathcal{F}$ from the category **FinRel** to the category of finite dimensional vector spaces over a field $\mathbb{K}$. It is defined as follows: On objects we let

$\mathcal{F}(X) := \mathcal{F}(X,\mathbb{K}) =$ the space of all $\mathbb{K}$-valued functions on X.

If $\Gamma \subset X \times Y$ is a relation and $g \in \mathcal{F}(Y)$ we set

$$(\mathcal{F}(\Gamma)(g))(x) := \sum_{y|(x,y)\in\Gamma} g(y) \ \ \forall\, x \in X.$$

(It is understood that the empty sum gives zero.) It is immediate to check that this is indeed a contravariant functor.

In case $\mathbb{K} = \mathbb{C}$ we can be more precise: Let us make $\mathcal{F}(X)$ into a (finite dimensional) Hilbert space by setting

$$(f_1,f_2) := \sum_{x\in X} f_1(x)\overline{f_2(x)}.$$

Then for $\Gamma \in \mathrm{Morph}(X,Y), f \in \mathcal{F}(X), g \in \mathcal{F}(Y)$ we have

$$(f,\mathcal{F}(\Gamma)g) = \sum_{(x,y)\in\Gamma} f(x)\overline{g(y)} = (\mathcal{F}(\Gamma^\dagger)f,g).$$

So

$$\mathcal{F}(\Gamma^\dagger) = \mathcal{F}(\Gamma)^\dagger,$$

i.e.

$$\mathcal{F} \circ \dagger = \dagger \circ \mathcal{F}$$

where the † on the left is the involution on **FinRel** and the † on the right is the operation carrying a linear transformation between Hilbert spaces into its adjoint. Thus the functor $\mathcal{F}$ carries the involutive structure of the category of finite sets and relations into the involutive structure of the category of finite dimensional Hilbert spaces.

3.3.5 Enhancing the category of finite sets and relations.

By a vector bundle over a finite set we simply mean a rule which assigns a vector space E_x (which we will assume to be finite dimensional) to each point x of X. We are going to consider a category whose objects are vector bundles over finite sets. We will denote such an object by $E \to X$.

Following Atiyah and Bott, we will define the morphisms in this category as follows: If $E \to X$ and $F \to Y$ are objects in our category, and $\Gamma \subset X \times Y$ we consider the vector bundle over Γ which assigns to each point $(x, y) \in \Gamma$ the vector space $\mathrm{Hom}(F_y, E_x)$. A morphism in our category will be a section of this vector bundle. So a morphism in our category will be a subset Γ of $X \times Y$ together with a map

$$r_{x,y} : F_y \to E_x$$

given for each $(x, y) \in \Gamma$.

Suppose that $(\Gamma_1, r) \in \mathrm{Morph}(E \to X, F \to Y)$ and $(\Gamma_2, s) \in \mathrm{Morph}(F \to Y, G \to Z)$. Their composition is defined to be $(\Gamma_2 \circ \Gamma_1, t)$ where t is the section of the vector bundle over $\Gamma_2 \circ \Gamma_1$ given by

$$t(x, z) = \sum_{y|(x,y)\in\Gamma_1,(y,z)\in\Gamma_2} r(x, y) \circ s(y, z).$$

The verification of the category axioms is immediate.

We have **enhanced** the category of finite sets and relations to the category of vector bundles over finite sets.

We also have a generalization of the functor $\mathcal{F}$: we now define $\mathcal{F}(E \to X)$ to be the space of sections of the vector bundle $E \to X$ and if $M \in \mathrm{Morph}(E \to X, F \to Y)$ then

$$\mathcal{F}(g)(x) = \sum_{y|(x,y)\in\Gamma} r(x,y)g(y).$$

This generalizes the Radon functor of the preceding section.

3.4 The linear symplectic category.

Let V_1 and V_2 be symplectic vector spaces with symplectic forms ω_1 and ω_2. We will let V_1^- denote the vector space V_1 equipped with the symplectic form $-\omega_1$. So $V_1^- \oplus V_2$ denotes the vector space $V_1 \oplus V_2$ equipped with the symplectic form $-\omega_1 \oplus \omega_2$.

A Lagrangian subspace Γ of $V_1^- \oplus V_2$ is called a **linear canonical relation**. The purpose of this section is to show that if we take the collection of symplectic vector spaces as objects, and the linear canonical relations as morphisms we get a category. Here composition is in the sense of composition of relations as in the category **FinRel**. In more detail: Let V_3 be a third symplectic vector space, let

$$\Gamma_1 \text{ be a Lagrangian subspace of } \quad V_1^- \oplus V_2$$

and let

$$\Gamma_2 \text{ be a Lagrangian subspace of } \quad V_2^- \oplus V_3.$$

Recall that as a *set* (see (3.1)) the composition

$$\Gamma_2 \circ \Gamma_1 \subset V_1 \times V_3$$

is defined by

$$(x,z) \in \Gamma_2 \circ \Gamma_1 \;\Leftrightarrow\; \exists\, y \in V_2 \text{ such that } (x,y) \in \Gamma_1 \text{ and } (y,z) \in \Gamma_2.$$

We must show that this is a Lagrangian subspace of $V_1^- \oplus V_3$. It will be important for us to break up the definition of $\Gamma_2 \circ \Gamma_1$ into two steps:

3.4.1 The space $\Gamma_2 \star \Gamma_1$.

Define

$$\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$$

to consist of all pairs $((x, y), (y', z))$ such that $y = y'$. We will restate this definition in two convenient ways. Let

$$\pi : \Gamma_1 \to V_2, \qquad \pi(v_1, v_2) = v_2$$

and

$$\rho : \Gamma_2 \to V_2, \qquad \rho(v_2, v_3) = v_2.$$

Let

$$\tau : \ \Gamma_1 \times \Gamma_2 \to V_2$$

be defined by

$$\tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

Then $\Gamma_2 \star \Gamma_1$ is determined by the exact sequence

$$0 \to \Gamma_2 \star \Gamma_1 \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \to \text{Coker } \tau \to 0. \quad (3.6)$$

Another way of saying the same thing is to use the language of "fiber products" or "exact squares". Let $f : A \to C$ and $g : B \to C$ be maps, say between sets. Then we express the fact that $F \subset A \times B$ consists of those pairs (a, b) such that $f(a) = g(b)$ by saying that

$$\begin{array}{ccc} F & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow f \\ B & \underset{g}{\longrightarrow} & C \end{array}$$

is an **exact square** or a **fiber product diagram**.

Thus another way of expressing the definition of $\Gamma_2 \star \Gamma_1$ is to say that

$$\begin{array}{ccc} \Gamma_2 \star \Gamma_1 & \longrightarrow & \Gamma_1 \\ \Big\downarrow & & \Big\downarrow \pi \\ \Gamma_2 & \underset{\rho}{\longrightarrow} & V_2 \end{array} \qquad (3.7)$$

is an exact square.

3.4.2 The transpose.

If $\Gamma \subset V_1^- \oplus V_2$ is a linear canonical relation, we define its transpose $\Gamma^\dagger$ just as in **FinRel**:

$$\Gamma^\dagger := \{(y,x)|(x,y) \in \Gamma\}. \tag{3.8}$$

Here $x \in V_1$ and $y \in V_2$ so $\Gamma^\dagger$ as defined is a linear Lagrangian subspace of $V_2 \oplus V_1^-$. But replacing the symplectic form by its negative does not change the set of Lagrangian subspaces, so $\Gamma^\dagger$ is also a Lagrangian subspace of $V_2^- \oplus V_1$, i.e. a linear canonical relation between V_2 and V_1. It is also obvious that just as in **FinRel** we have

$$\left(\Gamma^\dagger\right)^\dagger = \Gamma.$$

3.4.3 The projection $\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$.

Consider the map

$$\alpha : (x,y,y,z) \mapsto (x,z). \tag{3.9}$$

By definition

$$\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1.$$

3.4.4 The kernel and image of a linear canonical relation.

Let V_1 and V_2 be symplectic vector spaces and let $\Gamma \subset V_1^- \times V_2$ be a linear canonical relation. Let

$$\pi : \Gamma \to V_2$$

be the projection onto the second factor. Define

- $\operatorname{Ker}\Gamma \subset V_1$ by $\operatorname{Ker}\Gamma = \{v \in V_1 | (v,0) \in \Gamma\}$.
- $\operatorname{Im}\Gamma \subset V_2 = \pi(\Gamma) = \{v_2 \in V_2 | \exists v_1 \in V_1 \text{ with } (v_1,v_2) \in \Gamma\}$.

Now $\Gamma^\dagger \subset V_2^- \oplus V_1$ and hence both $\ker\Gamma^\dagger$ and $\operatorname{Im}\Gamma$ are linear subspaces of the symplectic vector space V_2. We claim that

$$(\ker\Gamma^\dagger)^\perp = \operatorname{Im}\Gamma. \tag{3.10}$$

Here $\perp$ means perpendicular relative to the symplectic structure on V_2.

Proof. Let ω_1 and ω_2 be the symplectic bilinear forms on V_1 and V_2 so that $\tilde{\omega} = -\omega_1 \oplus \omega_2$ is the symplectic form on $V_1^- \oplus V_2$. So $v \in V_2$ is in $\operatorname{Ker} \Gamma^\dagger$ if and only if $(0, v) \in \Gamma$. Since Γ is Lagrangian, $(0, v) \in \Gamma \Leftrightarrow (0, v) \in \Gamma^\perp$ and

$$(0, v) \in \Gamma^\perp \Leftrightarrow 0 = -\omega_1(0, v_1) + \omega_2(v, v_2) = \omega_2(v, v_2) \; \forall \, (v_1, v_2) \in \Gamma.$$

But this is precisely the condition that $v \in (\operatorname{Im} \Gamma)^\perp$. □

The kernel of α consists of those $(0, v, v, 0) \in \Gamma_2 \star \Gamma_1$. We may thus identify

$$\ker \alpha = \ker \Gamma_1^\dagger \cap \ker \Gamma_2 \tag{3.11}$$

as a subspace of V_2.

If we go back to the definition of the map τ, we see that the image of τ is given by

$$\operatorname{Im} \tau = \operatorname{Im} \Gamma_1 + \operatorname{Im} \Gamma_2^\dagger, \tag{3.12}$$

a subspace of V_2. If we compare (3.11) with (3.12) we see that

$$\ker \alpha = (\operatorname{Im} \tau)^\perp \tag{3.13}$$

as subspaces of V_2 where $\perp$ denotes orthocomplement relative to the symplectic form ω_2 of V_2.

3.4.5 Proof that $\Gamma_2 \circ \Gamma_1$ is Lagrangian.

Since $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$ and $\Gamma_2 \star \Gamma_1 = \ker \tau$ it follows that $\Gamma_2 \circ \Gamma_1$ is a linear subspace of $V_1^- \oplus V_3$.

It is equally easy to see that $\Gamma_2 \circ \Gamma_1$ is an isotropic subspace of $V_1^- \oplus V_2$. Indeed, if (x, z) and (x', z') are elements of $\Gamma_2 \circ \Gamma_1$, then there are elements y and y' of V_2 such that

$$(x, y) \in \Gamma_1, \; (y, z) \in \Gamma_2, \; (x', y') \in \Gamma_1, \; (y', z') \in \Gamma_2.$$

Then

$$\omega_3(z, z') - \omega_1(x, x') = \omega_3(z, z') - \omega_2(y, y') + \omega_2(y, y') - \omega_1(x, x') = 0.$$

So we must show that $\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$. It follows from (3.13) that

$$\dim \; \ker \; \alpha = \dim \; V_2 - \dim \operatorname{Im} \; \tau$$

and from the fact that $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$ that

$$\dim\ \Gamma_2 \circ \Gamma_1 = \dim\ \Gamma_2 \star \Gamma_1 - \dim\ \ker \alpha =$$

$$= \dim\ \Gamma_2 \star \Gamma_1 - \dim\ V_2 + \dim \mathrm{Im}\ \tau.$$

Since $\Gamma_2 \star \Gamma_1$ is the kernel of the map $\tau : \Gamma_1 \times \Gamma_2 \to V_2$ it follows that

$$\dim\ \Gamma_2 \star \Gamma_1 = \dim \Gamma_1 \times \Gamma_2 - \dim\ \mathrm{Im}\ \tau =$$

$$\frac{1}{2} \dim\ V_1 + \frac{1}{2} \dim\ V_2 + \frac{1}{2} \dim\ V_2 + \frac{1}{2} \dim\ V_3 - \dim\ \mathrm{Im}\ \tau.$$

Putting these two equations together we see that

$$\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$$

as desired. We have thus proved

Theorem 6 *The composite* $\Gamma_2 \circ \Gamma_1$ *of two linear canonical relations is a linear canonical relation.*

The diagonal Δ_V gives the identity morphism and so we have verified that **LinSym** is a category whose objects are symplectic vector spaces and whose morphisms are linear canonical relations.

3.4.6 The category LinSym and the symplectic group.

The category **LinSym** is a vast generalization of the symplectic group because of the following observation: Let X and Y be symplectic vector spaces. Suppose that the Lagrangian subspace $\Gamma \subset X^- \oplus Y$ projects bijectively onto X under the projection of $X \oplus Y$ onto the first factor. This means that Γ is the graph of a linear transformation T from X to Y:

$$\Gamma = \{(x, Tx)\}.$$

T must be injective. Indeed, if $Tx = 0$ the fact that Γ is isotropic implies that $x \perp X$ so $x = 0$. Also T is surjective since if $y \perp \mathrm{im}(T)$, then $(0, y) \perp \Gamma$. This implies that $(0, y) \in \Gamma$ since Γ is maximal isotropic. By the bijectivity of the projection of Γ onto X, this implies that $y = 0$. In other words T is a bijection. The fact that Γ is isotropic then says that

$$\omega_Y(Tx_1, Tx_2) = \omega_X(x_1, x_2),$$

i.e. T is a symplectic isomorphism. If $\Gamma_1 = \text{graph}\, T$ and $\Gamma_2 = \text{graph}\, S$ then

$$\Gamma_2 \circ \Gamma_1 = \text{graph}\, S \circ T$$

so composition of Lagrangian relations reduces to composition of symplectic isomorphisms in the case of graphs. In particular, if we take $Y = X$ we see that $\text{Symp}(X)$ is a subgroup of Morph (X, X) in our category.

Chapter 4

The Symplectic "Category".

Let M be a symplectic manifold with symplectic form ω. Then $-\omega$ is also a symplectic form on M. We will frequently write M instead of (M,ω) and by abuse of notation we will let M^- denote the manifold M with the symplectic form $-\omega$.

Let (M_i,ω_i) $i=1,2$ be symplectic manifolds. A Lagrangian submanifold Γ of

$$\Gamma \subset M_1^- \times M_2$$

is called a **canonical relation**. So Γ is a subset of $M_1 \times M_2$ which is a Lagrangian submanifold relative to the symplectic form $\omega_2 - \omega_1$ in the obvious notation. So a canonical relation is a relation which is a Lagrangian submanifold.

For example, if $f : M_1 \to M_2$ is a symplectomorphism, then $\Gamma_f = \text{graph } f$ is a canonical relation.

If $\Gamma_1 \subset M_1 \times M_2$ and $\Gamma_2 \subset M_2 \times M_3$ we can form their composite

$$\Gamma_2 \circ \Gamma_1 \subset M_1 \times M_3$$

in the sense of the composition of relations. So $\Gamma_2 \circ \Gamma_1$ consists of all points (x,z) such that there exists a $y \in M_2$ with $(x,y) \in \Gamma_1$ and $(y,z) \in \Gamma_2$.

Let us put this in the language of fiber products: Let

$$\pi : \Gamma_1 \to M_2$$

denote the restriction to Γ_1 of the projection of $M_1 \times M_2$ onto the second factor. Let

$$\rho : \Gamma_2 \rightarrow M_2$$

denote the restriction to Γ_2 of the projection of $M_2 \times M_3$ onto the first factor. Let

$$F \subset M_1 \times M_2 \times M_2 \times M_3$$

be defined by

$$F = (\pi \times \rho)^{-1} \Delta_{M_2}.$$

In other words, F is defined as the fiber product (or exact square)

$$\begin{array}{ccc} F & \xrightarrow{\iota_1} & \Gamma_1 \\ {\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \pi}. \\ \Gamma_2 & \xrightarrow[\rho]{} & M_2 \end{array} \qquad (4.1)$$

so

$$F \subset \Gamma_1 \times \Gamma_2 \subset M_1 \times M_2 \times M_2 \times M_3.$$

Let pr_{13} denote the projection of $M_1 \times M_2 \times M_2 \times M_3$ onto $M_1 \times M_3$ (projection onto the first and last components). Let π_{13} denote the restriction of pr_{13} to F. Then, as a *set*,

$$\Gamma_2 \circ \Gamma_1 = \pi_{13}(F). \qquad (4.2)$$

The map pr_{13} is smooth, and hence its restriction to any submanifold is smooth. The problems are that

1. F defined as

$$F = (\pi \times \rho)^{-1} \Delta_{M_2},$$

   i.e. by (4.1), need not be a submanifold, and

2. that the restriction π_{13} of pr_{13} to F need not be an embedding.

So we need some additional hypotheses to ensure that $\Gamma_2 \circ \Gamma_1$ is a submanifold of $M_1 \times M_3$. Once we impose these hypotheses we will find it easy to check that $\Gamma_2 \circ \Gamma_1$ is a Lagrangian submanifold of $M_1^- \times M_3$ and hence a canonical relation.

4.1 Clean intersection.

Assume that the maps

$$\pi : \Gamma_1 \to M_2 \quad \text{and} \quad \rho : \Gamma_2 \to M_2$$

defined above intersect cleanly.

Notice that $(m_1, m_2, m_2', m_3) \in F$ if and only if

- $m_2 = m_2'$,
- $(m_1, m_2) \in \Gamma_1$, and
- $(m_2', m_3) \in \Gamma_2$.

So we can think of F as the subset of $M_1 \times M_2 \times M_3$ consisting of all points (m_1, m_2, m_3) with $(m_1, m_2) \in \Gamma_1$ and $(m_2, m_3) \in \Gamma_2$. The clean intersection hypothesis involves two conditions. The first is that F be a manifold. The second is that the derived square be exact at all points. Let us state this second condition more explicitly: Let $m = (m_1, m_2, m_3) \in F$. We have the following vector spaces:

$$\begin{aligned} V_1 &:= T_{m_1} M_1, \\ V_2 &:= T_{m_2} M_2, \\ V_3 &:= T_{m_3} M_3, \\ \Gamma_1^m &:= T_{(m_1,m_2)} \Gamma_1, \quad \text{and} \\ \Gamma_2^m &:= T_{(m_2,m_3)} \Gamma_2. \end{aligned}$$

So

$$\Gamma_1^m \subset T_{(m_1,m_2)}(M_1 \times M_2) = V_1 \oplus V_2$$

is a linear Lagrangian subspace of $V_1^- \oplus V_2$. Similarly, Γ_2^m is a linear Lagrangian subspace of $V_2^- \oplus V_3$. The *clean intersection hypothesis* asserts that $T_m F$ is given by the exact square

$$\begin{array}{ccc} T_m F & \xrightarrow{d(\iota_1)_m} & \Gamma_1^m \\ {\scriptstyle d(\iota_2)_m}\downarrow & & \downarrow {\scriptstyle d\pi_{(m_1,m_2)}} \\ \Gamma_2^m & \xrightarrow[d\rho_{(m_2,m_3)}]{} & T_{m_2} M_2 \end{array} \tag{4.3}$$

In other words, $T_m F$ consists of all $(v_1, v_2, v_3) \in V_1 \oplus V_2 \oplus V_3$ such that

$$(v_1, v_2) \in \Gamma_1^m \quad \text{and} \quad (v_2, v_3) \in \Gamma_2^m.$$

The exact square (4.3) is of the form (3.7) that we considered in Section 3.4. We know from Section 3.4 that $\Gamma_2^m \circ \Gamma_1^m$ is a linear Lagrangian subspace of $V_1^- \oplus V_3$. In particular its dimension is $\frac{1}{2}(\dim M_1 + \dim M_3)$ which does not depend on the choice of $m \in F$. This implies the following: Let

$$\iota : F \to M_1 \times M_2 \times M_3$$

denote the inclusion map, and let

$$\kappa_{13} : M_1 \times M_2 \times M_3 \to M_1 \times M_3$$

denote the projection onto the first and third components. So

$$\kappa_{13} \circ \iota : F \to M_1 \times M_3$$

is a smooth map whose differential at any point $m \in F$ maps $T_m F$ onto $\Gamma_2^m \circ \Gamma_1^m$ and so has locally constant rank. Furthermore, the image of $T_m F$ is a Lagrangian subspace of $T_{(m_1,m3)}(M_1^- \times M_3)$. We have proved:

Theorem 7 *If the canonical relations $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ intersect cleanly, then their composition $\Gamma_2 \circ \Gamma_1$ is an immersed Lagrangian submanifold of $M_1^- \times M_3$.*

We must still impose conditions that will ensure that $\Gamma_2 \circ \Gamma_1$ is an honest submanifold of $M_1 \times M_3$. We will do this in the next section.

We will need a name for the manifold F we created out of Γ_1 and Γ_2 above. As in the linear case, we will call it $\Gamma_2 \star \Gamma_1$.

4.2 Composable canonical relations.

We recall a theorem from differential topology:

Theorem 8 *Let X and Y be smooth manifolds and $f : X \to Y$ is a smooth map of constant rank. Let $W = f(X)$. Suppose that f is proper and that for every $w \in W$, $f^{-1}(w)$ is connected and simply connected. Then W is a smooth submanifold of Y.*

We apply this theorem to the map $\kappa_{13} \circ \iota : F \to M_1 \times M_3$. To shorten the notation, let us define

$$\kappa := \kappa_{13} \circ \iota. \tag{4.4}$$

Theorem 9 *Suppose that the canonical relations* Γ_1 *and* Γ_2 *intersect cleanly. Suppose in addition that the map* κ *is proper and that the inverse image of every* $\gamma \in \Gamma_2 \circ \Gamma_1 = \kappa(\Gamma_2 \star \Gamma_1)$ *is connected and simply connected. Then* $\Gamma_2 \circ \Gamma_1$ *is a canonical relation. Furthermore*

$$\kappa : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1 \tag{4.5}$$

is a smooth fibration with compact connected fibers.

So we are in the following situation: We can not always compose the canonical relations $\Gamma_2 \subset M_2^- \times M_3$ and $\Gamma_1 \subset M_1^- \times M_2$ to obtain a canonical relation $\Gamma_2 \circ \Gamma_1 \subset M_1^- \times M_3$. We must impose some additional conditions, for example those of the theorem. So, following Weinstein, we put quotation marks around the word category to indicate this fact.

We will let $\mathcal{S}$ denote the "category" whose objects are symplectic manifolds and whose morphisms are canonical ralations. We will call $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ **cleanly composable** if they satisfy the hypotheses of Theorem 9.

If $\Gamma \subset M_1^- \times M_2$ is a canonical relation, we will sometimes use the notation

$$\Gamma \in \mathrm{Morph}(M_1, M_2)$$

and sometimes use the notation

$$\Gamma : \; M_1 \twoheadrightarrow M_2$$

to denote this fact.

4.3 Transverse composition.

A special case of clean intersection is transverse intersection. In fact, in applications, this is a convenient hypothesis, and it has some special properties:

Suppose that the maps π and ρ are transverse. This means that

$$\pi \times \rho : \Gamma_1 \times \Gamma_2 \to M_2 \times M_2$$

intersects Δ_{M_2} transversally, which implies that the codimension of

$$\Gamma_2 \star \Gamma_1 = (\pi \times \rho)^{-1}(\Delta_{M_2})$$

in $\Gamma_1 \times \Gamma_2$ is $\dim M_2$. So with $F = \Gamma_2 \star \Gamma_1$ we have

$$\begin{aligned}
\dim F &= \dim \Gamma_1 + \dim \Gamma_2 - \dim M_2 \\
&= \frac{1}{2}\dim M_1 + \frac{1}{2}\dim M_2 + \frac{1}{2}\dim M_2 + \frac{1}{2}\dim M_3 - \dim M_2 \\
&= \frac{1}{2}\dim M_1 + \frac{1}{2}\dim M_3 \\
&= \dim \Gamma_2 \circ \Gamma_1.
\end{aligned}$$

So under the hypothesis of transversality, the map $\kappa = \kappa_{13} \circ \iota$ is an immersion. If we add the hypotheses of Theorem 9, we see that κ is a diffeomorphism.

For example, if Γ_2 is the graph of a symplectomorphism of M_2 with M_3 then $d\rho_{(m_2,m_3)} : T_{(m_2,m_3)}(\Gamma) \to T_{m_2}M_2$ is surjective at all points $(m_2, m_3) \in \Gamma_2$. So if $m = (m_1, m_2, m_2, m_3) \in \Gamma_1 \times \Gamma_2$ the image of $d(\pi \times \rho)_m$ contains all vectors of the form $(0, w)$ in $T_{m_2}M_2 \oplus T_{m_2}M_2$ and so is transverse to the diagonal. The manifold $\Gamma_2 \star \Gamma_1$ consists of all points of the form $(m_1, m_2, g(m_2))$ with $(m_1, m_2) \in \Gamma_1$, and

$$\kappa : (m_1, m_2, g(m_2)) \mapsto (m_1, g(m_2)).$$

Since g is one to one, so is κ. So the graph of a symplectomorphism is transversally composible with *any* canonical relation.

We will need the more general concept of “clean composability” described in the preceding section for certain applications.

4.4 Lagrangian submanifolds as canonical relations.

We can consider the “zero dimensional symplectic manifold” consisting of the distinguished point that we call “pt.”. Then a canonical relation between pt. and a symplectic manifold M is a Lagrangian submanifold of pt. $\times M$ which may be identified with a Lagrangian submanifold of M. These are the “points” in our “category” $\mathcal{S}$.

Suppose that Λ is a Lagrangian submanifold of M_1 and $\Gamma \in \mathrm{Morph}(M_1, M_2)$ is a canonical relation. If we think of Λ as an element of $\mathrm{Morph}(\mathrm{pt.}, M_1)$, then if Γ and Λ are composible, we can form $\Gamma \circ \Lambda \in \mathrm{Morph}(\mathrm{pt.}, M_2)$ which may be identified with a Lagrangian submanifold of M_2. If we want to think of it this way, we may sometimes write $\Gamma(\Lambda)$ instead of $\Gamma \circ \Lambda$.

We can mimic the construction of composition given in Section 3.3.2 for the category of finite sets and relations. Let M_1, M_2 and M_3 be symplectic manifolds and let $\Gamma_1 \in \mathrm{Morph}(M_1, M_2)$ and $\Gamma_2 \in \mathrm{Morph}(M_2, M_3)$ be canonical relations. So

$$\Gamma_1 \times \Gamma_2 \subset M_1^- \times M_2 \times M_2^- \times M_3$$

is a Lagrangian submanifold. Let

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x,y,y,z,x,z)\} \subset M_1 \times M_2 \times M_2 \times M_3 \times M_1 \times M_3. \tag{4.6}$$

We endow the right hand side with the symplectic structure

$$M_1 \times M_2^- \times M_2 \times M_3^- \times M_1^- \times M_3 = (M_1^- \times M_2 \times M_2^- \times M_3)^- \times (M_1^- \times M_3).$$

Then $\tilde{\Delta}_{M_1,M_2,M_3}$ is a Lagrangian submanifold, i.e. an element of

$$\mathrm{Morph}(M_1^- \times M_2 \times M_2^- \times M_3, M_1^- \times M_3).$$

Just as in Section 3.3.2,

$$\tilde{\Delta}_{M_1,M_2,M_3}(\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1.$$

It is easy to check that Γ_2 and Γ_1 are composible if and only if $\tilde{\Delta}_{M_1,M_2,M_3}$ and $\Gamma_1 \times \Gamma_2$ are composible.

4.5 The involutive structure on $\mathcal{S}$.

Let $\Gamma \in \mathrm{Morph}(M_1, M_2)$ be a canonical relation. Just as in the category of finite sets and relations, define

$$\Gamma^\dagger = \{(m_2, m_1) | (m_1, m_2) \in \Gamma\}.$$

As a set it is a subset of $M_2 \times M_1$ and it is a Lagrangian submanifold of $M_2 \times M_1^-$. But then it is also a Lagrangian submanifold of

$$(M_2 \times M_1^-)^- = M_2^- \times M_1.$$

So

$$\Gamma^\dagger \in \mathrm{Morph}(M_2, M_1).$$

Therefore $M \mapsto M, \Gamma \mapsto \Gamma^\dagger$ is a involutive functor on $\mathcal{S}$.

4.6 Canonical relations between cotangent bundles.

In this section we want to discuss some special properties of our “category” $\mathcal{S}$ when we restrict the objects to be cotangent bundles (which are, after all, special kinds of symplectic manifolds). One consequence of our discussion will be that $\mathcal{S}$ contains the category $\mathcal{C}^\infty$ whose objects are smooth manifolds and whose morphisms are smooth maps as a (tiny) subcategory. Another consequence will be a local description of Lagrangian submanifolds of the cotangent bundle which generalizes the description of horizontal Lagrangian submanifolds of the cotangent bundle that we gave in Chapter 1. We will use this local description to deal with the problem of passage through caustics that we encountered in Chapter 1.

We recall the following definitions from Chapter 1: Let X be a smooth manifold and T^*X its cotangent bundle, so that we have the projection $\pi : T^*X \to X$. The canonical one form α_X is defined by (1.8). We repeat the definition: If $\xi \in T^*X, x = \pi(\xi)$, and $v \in T_\xi(T^*X)$ then the value of α_X at v is given by

$$\langle \alpha_X, v\rangle := \langle \xi, d\pi_\xi v\rangle. \tag{1.8}$$

The symplectic form ω_X is given by

$$\omega_X = -d\alpha_X. \tag{1.10}$$

So if Λ is a submanifold of T^*X on which α_X vanishes and whose dimension is $\dim X$ then Λ is (a special kind of) Lagrangian submanifold of T^*X. An instance of this is the conormal bundle of a submanifold: Let $Y \subset X$ be a submanifold. Its conormal bundle

$$N^*Y \subset T^*X$$

consists of all $(x, \xi) \in T^*X$ such that $x \in Y$ and ξ vanishes on T_xY. If $v \in T_\xi(N^*Y)$ then $d\pi_\xi(v) \in T_xY$ so by (1.8) $\langle \alpha_X, v\rangle = 0$.

4.7 The canonical relation associated to a map.

Let X_1 and X_2 be manifolds and $f : X_1 \to X_2$ be a smooth map. We set

$$M_1 := T^*X_1 \quad \text{and} \quad M_2 := T^*X_2$$

with their canonical symplectic structures. We have the identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 = T^*(X_1 \times X_2).$$

The graph of f is a submanifold of $X_1 \times X_2$:

$$X_1 \times X_2 \supset \text{graph}(f) = \{(x_1, f(x_1))\}.$$

So the conormal bundle of the graph of f is a Lagrangian submanifold of $M_1 \times M_2$. Explicitly,

$$N^*(\text{graph}(f)) = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1),\ \ \xi_1 = -df^*_{x_1}\xi_2\}. \tag{4.7}$$

Let

$$\varsigma_1 : T^*X_1 \to T^*X_1$$

be defined by

$$\varsigma_1(x, \xi) = (x, -\xi).$$

Then $\varsigma_1^*(\alpha_{X_1}) = -\alpha_{X_1}$ and hence

$$\varsigma_1^*(\omega_{X_1}) = -\omega_{X_1}.$$

We can think of this as saying that ς_1 is a symplectomorphism of M_1 with M_1^- and hence

$$\varsigma_1 \times \text{id}$$

is a symplectomorphism of $M_1 \times M_2$ with $M_1^- \times M_2$. Let

$$\Gamma_f := (\varsigma_1 \times \text{id})(N^*(\text{graph}(f))). \tag{4.8}$$

Then Γ_f is a Lagrangian submanifold of $M_1^- \times M_2$. In other words,

$$\Gamma_f \in \text{Morph}(M_1, M_2).$$

Explicitly,

$$\Gamma_f = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1), \quad \xi_1 = df^*_{x_1}\xi_2\}. \tag{4.9}$$

Suppose that $g : X_2 \to X_3$ is a smooth map so that $\Gamma_g \in \mathrm{Morph}(M_2, M_3)$. So

$$\Gamma_g = \{(x_2, \xi_2, x_3, \xi_3) | x_3 = g(x_2), \xi_2 = dg^*_{x_2}\xi_3.\}.$$

The maps

$$\pi : \Gamma_f \to M_2, \qquad (x_1, \xi_1, x_2, \xi_2) \mapsto (x_2, \xi_2)$$

and

$$\rho : \Gamma_g \to M_2, \qquad (x_2, \xi_2, x_3, \xi_3) \mapsto (x_2, \xi_2)$$

are transverse. Indeed at any point $(x_1, \xi_1, x_2, \xi_2, x_2, \xi_2, x_3, \xi_3)$ the image of $d\pi$ contains all vectors of the form $(0, w)$ in $T_{x_2,\xi_2}(T^*M_2)$, and the image of $d\rho$ contains all vectors of the form $(v, 0)$. So Γ_g and Γ_f are transversely composible. Their composite $\Gamma_g \circ \Gamma_f$ consists of all (x_1, ξ_1, x_3, ξ_3) such that there exists an x_2 such that $x_2 = f(x_1)$ and $x_3 = g(x_2)$ and a ξ_2 such that $\xi_1 = df^*_{x_1}\xi_2$ and $\xi_2 = dg^*_{x_2}\xi_3$. But this is precisely the condition that $(x_1, \xi_1, x_3, \xi_3) \in \Gamma_{g\circ f}$! We have proved:

Theorem 10 *The assignments*

$$X \mapsto T^*X$$

and

$$f \mapsto \Gamma_f$$

define a covariant functor from the category $\mathcal{C}^\infty$ of manifolds and smooth maps to the symplectic "category" $\mathcal{S}$. As a consequence the assignments $X \mapsto T^*X$ and

$$f \mapsto (\Gamma_f)^\dagger$$

define a contravariant functor from the category $\mathcal{C}^\infty$ of manifolds and smooth maps to the symplectic "category" $\mathcal{S}$.

We now study special cases of these functors in a little more detail:

4.8 Pushforward of Lagrangian submanifolds of the cotangent bundle.

Let $f : X_1 \to X_2$ be a smooth map, and $M_1 := T^*X_1$, $M_2 := T^*X_2$ as before. The Lagrangian submanifold $\Gamma_f \subset M_1^- \times M_2$ is defined by (4.9). In particular, it is a subset of $T^*X_1 \times T^*X_2$ and hence a particular kind of relation (in the sense of Chapter 3). So if A is any subset of T^*X_1 then $\Gamma_f(A)$ is a subset of T^*X_2 which we shall also denote by $df_*(A)$. So

$$df_*(A) := \Gamma_f(A), \qquad A \subset T^*X_1.$$

Explicitly,

$$df_*A = \{(y,\eta) \in T^*X_2 | \exists (x,\xi) \in A \text{ with } y = f(x) \text{ and } \xi = df_x^*\eta\}.$$

Now suppose that $A = \Lambda$ is a Lagrangian submanifold of T^*X_1. Considering Λ as an element of $\mathrm{Morph}(\mathrm{pt.}, T^*X_1)$ we may apply Theorem 7. Let

$$\pi_1 : \; N^*(\mathrm{graph}(f)) \to T^*X_1$$

denote the restriction to $N^*(\mathrm{graph}(f))$ of the projection of $T^*X_1 \times T^*X_2$ onto the first component. Notice that $N^*(\mathrm{graph}(f))$ is stable under the map $(x,\xi,y,\eta) \mapsto (x,-\xi,y,-\eta)$ and hence π_1 intersects Λ cleanly if and only if $\pi_1 \circ (\varsigma \times \mathrm{id}) : \Gamma_f \to T^*X_1$ intersects Λ cleanly where, by abuse of notation, we have also denoted by π_1 restriction of the projection to Γ_f. So

Theorem 11 *If Λ is a Lagrangian submanifold and $\pi_1 : N^*(\mathrm{graph}(f)) \to T^*X_1$ intersects Λ cleanly then $df_*(\Lambda)$ is an immersed Lagrangian submanifold of T^*X_2.*

If f has constant rank, then the dimension of $df_x^* T^*(X_2)_{f(x)}$ does not vary, so that $df^*(T^*X_2)$ is a sub-bundle of T^*X_1. If Λ intersects this subbundle transversally, then our conditions are certainly satisified. So

Theorem 12 *Suppose that $f : X_1 \to X_2$ has constant rank. If Λ is a Lagrangian submanifold of T^*X_1 which intersects $df^*T^*X_2$ transversaly then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

For example, if f is an immersion, then $df^*T^*X_2 = T^*X_1$ so all Lagrangian submanifolds are transverse to $df^*T^*X_2$.

Corollary 13 *If f is an immersion, then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

At the other extreme, suppose that $f : X_1 \to X_2$ is a fibration. Then $H^*(X_1) := df^*T^*N$ consists of the "horizontal sub-bundle", i.e those covectors which vanish when restricted to the tangent space to the fiber. So

Corollary 14 *Let $f : X_1 \to X_2$ be a fibration, and let $H^*(X_1)$ be the bundle of the horizontal covectors in T^*X_1. If Λ is a Lagrangian submanifold of T^*X_1 which intersects $H^*(X_1)$ transversaly, then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

An important special case of this corollary for us will be when $\Lambda = \text{graph } d\phi$. Then $\Lambda \cap H^*(X_1)$ consists of those points where the "vertical derivative", i.e. the derivative in the fiber direction vanishes. At such points $d\phi$ descends to give a covector at $x_2 = f(x_1)$. If the intersection is transverse, the set of such covectors is then a Lagrangian submanifold of T^*N. All of the next chapter will be devoted to the study of this special case of Corollary 14.

4.8.1 Envelopes.

Another important special case of Corollary 14 is the theory of envelopes, a classical subject which has more or less disappeared from the standard curriculum:

Let

$$X_1 = X \times S, \qquad X_2 = X$$

where X and S are manifolds and let $f = \pi : X \times S \to X$ be projection onto the first component.

Let

$$\phi : X \times S \to \mathbb{R}$$

be a smooth function having 0 as a regular value so that

$$Z := \phi^{-1}(0)$$

is a submanifold of $X \times S$. In fact, we will make a stronger assumption: Let $\phi_s : X \to \mathbb{R}$ be the map obtained by holding s fixed:

$$\phi_s(x) := \phi(x, s).$$

We make the stronger assumption that each ϕ_s has 0 as a regular value, so that

$$Z_s := \phi_s^{-1}(0) = Z \cap (X \times \{s\})$$

is a submanifold and

$$Z = \bigcup_s Z_s$$

as a set. The Lagrangian submanifold $N^*(Z) \subset T^*(X \times S)$ consists of all points of the form

$$(x, s, td\phi_X(x, s), td_S\phi(x, s)) \quad \text{such that } \phi(x, s) = 0.$$

Here t is an arbitrary real number. The sub-bundle $H^*(X \times S)$ consists of all points of the form

$$(x, s, \xi, 0).$$

So the transversality condition of Corollary 14 asserts that the map

$$z \mapsto d\left(\frac{\partial \phi}{\partial s}\right)$$

has rank equal to dim S on Z. The image Lagrangian submanifold $df_*N^*(Z)$ then consists of all covectors $td_X\phi$ where

$$\phi(x, s) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial s}(x, s) = 0,$$

a system of $p + 1$ equations in $n + p$ variables, where $p = \dim S$ and $n = \dim X$

Our transversality assumptions say that these equations define a submanifold of $X \times S$. If we make the stronger hypothesis that the last p equations can be solved for s as a function of x, then the first equation becomes

$$\phi(x, s(x)) = 0$$

which defines a hypersurface $\mathcal{E}$ called the **envelope** of the surfaces Z_s. Furthermore, by the chain rule,

$$d\phi(\cdot, s(\cdot)) = d_X\phi(\cdot, s(\cdot)) + d_S\phi(\cdot, s(\cdot))d_Xs(\cdot) = d_X\phi(\cdot, s(\cdot))$$

since $d_S\phi = 0$ at the points being considered. So if we set

$$\psi := \phi(\cdot, s(\cdot))$$

we see that under these restrictive hypotheses $df_*N^*(Z)$ consists of all multiples of $d\psi$, i.e.

$$df_*(N^*(Z)) = N^*(\mathcal{E})$$

is the normal bundle to the envelope.

In the classical theory, the envelope "develops singularities". But from our point of view it is natural to consider the Lagrangian submanifold $df_*N^*(Z)$. This will not be globally a normal bundle to a hypersurface because its projection on X (from T^*X) may have singularities. But as a submanifold of T^*X it is fine:

Examples:

- Suppose that S is an oriented curve in the plane, and at each point $s \in S$ we draw the normal ray to S at s. We might think of this line as a light ray propagating down the normal. The initial curve is called an "initial wave front" and the curve along which the light tends to focus is called the "caustic". Focusing takes place where "nearby normals intersect" i.e. at the envelope of the family of rays. These are the points which are the loci of the centers of curvature of the curve, and the corresponding curve is called the evolute.

- We can let S be a hypersurface in n-dimensions, say a surface in three dimensions. We can consider a family of lines emanating from a point source (possible at infinity), and reflected by S. The corresponding envelope is called the "caustic by reflection". In Descartes' famous theory of the rainbow he considered a family of parallel lines (light rays from the sun) which were refracted on entering a spherical raindrop, internally reflected by the opposite side and refracted again when exiting the raindrop. The corresponding "caustic" is the Descartes cone of 42 degrees.

- If S is a submanifold of $\mathbb{R}^n$ we can consider the set of spheres of radius r centered at points of S. The corresponding envelope consist of "all points at distance r from S". But this develops singularities past the radii of curvature. Again, from the Lagrangian or "upstairs" point of view there is no problem.

4.9 Pullback of Lagrangian submanifolds of the cotangent bundle.

We now investigate the contravariant functor which assigns to the smooth map $f : X_1 \to X_2$ the canonical relation

$$\Gamma_f^\dagger : \quad T^*X_2 \twoheadrightarrow T^*X_1.$$

As a subset of $T^*(X_2) \times T^*(X_1)$, $\Gamma_f^\dagger$ consists of all

$$(y, \eta, x, \xi)|\ y = f(x), \quad \text{and} \quad \xi = df_x^*(\eta). \tag{4.10}$$

If B is a subset of T^*X_2 we can form $\Gamma_f^\dagger(B) \subset T^*X_1$ which we shall denote by $df^*(B)$. So

$$df^*(B) := \Gamma_f^\dagger(B) = \{(x,\xi)|\exists\, b = (y,\eta) \in B \text{ with } f(x) = y,\ df_x^*\eta = \xi\}. \tag{4.11}$$

If $B = \Lambda$ is a Lagrangian submanifold, once again we may apply Theorem 7 to obtain a sufficient condition for $df^*(\Lambda)$ to be a Lagrangian submanifold of T^*X_1. Notice that in the description of $\Gamma_f^\dagger$ given in (4.10), the η can vary freely in $T^*(X_2)_{f(x)}$. So the issue of clean or transverse intersection comes down to the behavior of the first component. So, for example, we have the following theorem:

Theorem 15 *Let $f : X_1 \to X_2$ be a smooth map and Λ a Lagrangian submanifold of T^*X_2. If the maps f, and the restriction of the projection $\pi : T^*X_2 \to X_2$ to Λ are transverse, then $df^*\Lambda$ is a Lagrangian submanifold of T^*X_1.*

Here are two examples of the theorem:

- Suppose that Λ is a horizontal Lagrangian submanifold of T^*X_2. This means that restriction of the projection $\pi : T^*X_2 \to X_2$ to Λ is a diffeomorphism and so the transversality condition is satisfied for any f. Indeed, if $\Lambda = \Lambda_\phi$ for a smooth function ϕ on X_2 then

$$f^*(\Lambda_\phi) = \Lambda_{f^*\phi}.$$

- Suppose that $\Lambda = N^*(Y)$ is the normal bundle to a submanifold Y of X_2. The transversality condition becomes the condition that the map f

is transversal to Y. Then $f^{-1}(Y)$ is a submanifold of X_1. If $x \in f^{-1}(Y)$ and $\xi = df_x^*\eta$ with $(f(x), \eta) \in N^*(Y)$ then ξ vanishes when restricted to $T(f^{-1}(Y))$, i.e. $(x, \xi) \in \mathcal{N}(f^{-1}(S))$. More precisely, the transversality asserts that at each $x \in f^{-1}(Y)$ we have $df_x(T(X_1)_x) + TY_{f(x)} = T(X_2)_{f(x)}$ so

$$T(X_1)_x/T(f^{-1}(Y))_x \cong T(X_2)_{f(x))}/TY_{f(x)}$$

and so we have an isomorphism of the dual spaces

$$N_x^*(f^{-1}(Y)) \cong N^*f(x)(Y).$$

In short, the pullback of $N^*(Y)$ is $N^*(f^{-1}(Y))$.

4.10 The moment map.

In this section we show how to give a categorical generalization of the classical moment map for a Hamiltonian group action. We begin with a review of the classical theory.

4.10.1 The classical moment map.

In this section we recall the classical moment map, especially from Weinstein's point of view.

Let (M, ω) be a symplectic manifold, K a connected Lie group and τ an action of K on M preserving the symplectic form. From τ one gets an infinitesimal action

$$\delta\tau : \mathfrak{k} \to \mathrm{Vect}(M) \tag{4.12}$$

of the Lie algebra, $\mathfrak{k}$, of K, mapping $\xi \in \mathfrak{k}$ to the vector field, $\delta\tau(\xi) =: \xi_M$. Here ξ_M is the infinitesimal generator of the one parameter group

$$t \mapsto \tau_{\exp -t\xi}.$$

The minus sign is to guarantee that $\delta\tau$ is a Lie algebra homomorphism.

In particular, for $p \in M$, one gets from (4.12) a linear map,

$$d\tau_p : \mathfrak{k} \to T_pM\,, \quad \xi \to \xi_M(p)\,; \tag{4.13}$$

and from ω_p a linear isomorphism,

$$T_p \to T_p^* \quad v \to i(v)\omega_p\,; \tag{4.14}$$

which can be composed with (4.13) to get a linear map

$$\tilde{d\tau_p} : \mathfrak{k} \to T_p^*M\,. \tag{4.15}$$

Definition 1 *A K-equivariant map*

$$\phi : M \to \mathfrak{k}^* \tag{4.16}$$

is a **moment map***, if for every $p \in M$:*

$$d\phi_p : T_pM \to \mathfrak{k}^* \tag{4.17}$$

is the transpose of the map (4.15).

The property (4.17) determines $d\phi_p$ at all points p and hence determines ϕ up to an additive constant, $c \in (\mathfrak{k}^*)^K$ if M is connected. Thus, in particular, if K is semi-simple, the moment map, if it exists, is unique. As for the existence of ϕ, the duality of (4.15) and (4.17) can be written in the form

$$i(\xi_M)\omega = d\langle\phi, \xi\rangle \tag{4.18}$$

for all $\xi \in \mathfrak{k}$; and this shows that the vector field, ξ_M, has to be Hamiltonian. If K is compact the converse is true. A sufficient condition for the existence of ϕ is that each of the vector fields, ξ_M, be Hamiltonian. (See for instance, [**?**], § 26.) An equivalent formulation of this condition will be useful below:

Definition 2 *A symplectomorphism, $f : M \to M$ is* **Hamiltonian** *if there exists a family of symplectomorphisms, $f_t : M \to M$, $0 \leq t \leq 1$, depending smoothly on t with $f_0 = id_M$ and $f_1 = f$, such that the vector field*

$$v_t = f_t^{-1}\frac{df_t}{dt}$$

is Hamiltonian for all t.

It is easy to see that ξ_M is Hamiltonian for all $\xi \in \mathfrak{k}$ if and only if the symplectomorphism, τ_g, is exact for all $g \in K$.

Our goal in this section is to describe a generalized notion of moment mapping in which there are no group actions involved. First, however, we recall a very suggestive way of thinking about moment mappings and the “moment geometry” associated with moment mappings, due to Alan Weinstein, [**?**]. From the left action of K on T^*K one gets a trivialization

$$T^*K = K \times \mathfrak{k}^*$$

and via this trivialization a Lagrangian submanifold

$$\Gamma_\tau = \{(m, \tau_g m, g, \phi(m))\,;\, m \in M\,,\, g \in K\}\,,$$

of $M \times M^- \times T^*K$, which Weinstein calls *the moment Lagrangian*. He views this as a canonical relation between $M^- \times M$ and T^*K, i.e. as a morphism

$$\Gamma_\tau : M^- \times M \twoheadrightarrow T^*K\,.$$

4.10.2 Families of symplectomorphisms.

We now turn to the first stage of our generalization of the moment map, where the group action is replaced by a family of symplectomorphisms:

Let (M, ω) be a symplectic manifold, S an arbitrary manifold and f_s, $s \in S$, a family of symplectomorphisms of M depending smoothly on s . For $p \in M$ and $s_0 \in S$ let $g_{s_0,p} : S \to M$ be the map, $g_{s_0,p}(s) = f_s \circ f_{s_0}^{-1}(p)$. Composing the derivative of $g_{s_0,p}$ at s_0

$$(dg_{s_0,p})_{s_0} : T_{s_0}S \to T_pM \tag{4.19}$$

with the map (4.14) one gets a linear map

$$(\widetilde{dg_{s_0,p}})_{s_0} : T_{s_0}S \to T_p^*M\,. \tag{4.20}$$

Now let Φ be a map of $M \times S$ into T^*S which is compatible with the projection, $M \times S \to S$ in the sense

$$\begin{array}{ccc} M \times S & \xrightarrow{\Phi} & T^*S \\ & \searrow & \downarrow \\ & & S \end{array}$$

commutes; and for $s_0 \in S$ let

$$\Phi_{s_0} : M \to T_{s_0}^*S$$

be the restriction of Φ to $M \times \{s_0\}$.

Definition 3 *Φ is a* **moment map** *if, for all s_0 and p,*

$$(d\Phi_{s_0})_p : T_pM \to T^*_{s_0}S \tag{4.21}$$

is the transpose of the map (4.20).

We will prove below that a sufficient condition for the existence of Φ is that the f_s's be Hamiltonian; and, assuming that Φ exists, we will consider the analogue for Φ of Weinstein's moment Lagrangian,

$$\Gamma_\Phi = \{(m, f_s(m), \Phi(m,s))\,;\, m \in M\,,\, s \in S\}\,, \tag{4.22}$$

and ask if the analogue of Weinstein's theorem is true: Is (4.22) a Lagrangian submanifold of $M \times M^- \times T^*S$?

Equivalently consider the imbedding of $M \times S$ into $M \times M^- \times T^*S$ given by the map

$$G : M \times S \to M \times M^- \times T^*S\,,$$

where $G(m,s) = (m, f_s(m), \Phi(m,s))$. Is this a Lagrangian imbedding? The answer is "no" in general, but we will prove:

Theorem 16 *The pull-back by G of the symplectic form on $M \times M^- \times T^*S$ is the pull-back by the projection, $M \times S \to S$ of a closed two-form, μ, on S.*

If μ is exact, i.e., if $\mu = d\nu$, we can modify Φ by setting

$$\Phi_{\text{new}}(m,s) = \Phi_{\text{old}}(m,s) - \nu_s\,,$$

and for this modified Φ the pull-back by G of the symplectic form on $M \times M^- \times T^*S$ will be zero; so we conclude:

Theorem 17 *If μ is exact, there exists a moment map, $\Phi : M \times S \to T^*S$, for which Γ_Φ is Lagrangian.*

The following converse result is also true.

Theorem 18 *Let Φ be a map of $M \times S$ into T^*S which is compatible with the projection of $M \times S$ onto S. Then if Γ_Φ is Lagrangian, Φ is a moment map.*

Remarks:

1. A moment map with this property is still far from being unique; however, the ambiguity in the definition of Φ is now a *closed* one-form, $\nu \in \Omega^1(S)$.

2. if $[\mu] \neq 0$ there is a simple expedient available for making Γ_Φ Lagrangian. One can modify the symplectic structure of T^*S by adding to the standard symplectic form the pull-back of $-\mu$ to T^*S.

3. Let $\mathcal{G}_e$ be the group of Hamiltonian symplectomorphisms of M. Then for every manifold, S and smooth map

$$F : S \to \mathcal{G}_e$$

one obtains by the construction above a cohomology class $[\mu]$ which is a homotopy invariant of the mapping F.

4. For a smooth map $F : S \to \mathcal{G}_e$, there exists an analogue of the character Lagrangian. Think of Γ_Φ as a canonical relation or "map"

$$\Gamma_\Phi : M^- \times M \twoheadrightarrow T^*S$$

and define the character Lagrangian of F to be the image with respect to Γ_Φ of the diagonal in $M^- \times M$.

Our proof of the results above will be an illustration of the principle: the more general the statement of a theorem the easier it is to prove. We will first generalize these results by assuming that the f_s's are canonical relations rather than canonical transformations, i.e., are "maps" from M to M in Weinstein's sense. Next we will get rid of "maps" altogether and replace $M \times M^-$ by M itself and canonical relations by Lagrangian submanifolds of M.

4.10.3 The moment map in general.

Let (M, ω) be a symplectic manifold. Let Z, X and S be manifolds and suppose that

$$\pi : Z \to S$$

is a fibration with fibers diffeomorphic to X. Let

$$G : Z \to M$$

be a smooth map and let

$$g_s : Z_s \to M, \quad Z_s := \pi^{-1}(s)$$

denote the restriction of G to Z_s. We assume that

$$g_s \text{ is a Lagrangian embedding} \tag{4.23}$$

and let

$$\Lambda_s := g_s(Z_s) \tag{4.24}$$

denote the image of g_s. Thus for each $s \in S$, the restriction of G imbeds the fiber, $Z_s = \pi^{-1}(s)$, into M as the Lagrangian submanifold, Λ_s. Let $s \in S$ and $\xi \in T_sS$. For $z \in Z_s$ and $w \in T_zZ_s$ tangent to the fiber Z_s

$$dG_z w = (dg_s)_z w \in T_{G(z)}\Lambda_s$$

so dG_z induces a map, which by abuse of language we will continue to denote by dG_z

$$dG_z : T_zZ/T_zZ_s \to T_mM/T_m\Lambda, \quad m = G(z). \tag{4.25}$$

But $d\pi_z$ induces an identification

$$T_zZ/T_z(Z_s) = T_sS. \tag{4.26}$$

Furthermore, we have an identification

$$T_mM/T_m(\Lambda_s) = T_m^*\Lambda_s \tag{4.27}$$

given by

$$T_mM \ni u \mapsto i(u)\omega_m(\cdot) = \omega_m(u, \cdot).$$

Finally, the diffeomorphism $g_s : Z_s \to \Lambda_s$ allows us to identify

$$T_m^*\Lambda_s \sim T_z^*Z_s, \quad m = G(z).$$

Via all these identifications we can convert (4.25) into a map

$$T_sS \to T_z^*Z_s\,. \tag{4.28}$$

Now let $\Phi : Z \to T^*S$ be a lifting of $\pi : Z \to S$, so that

$$\begin{array}{ccc} Z & \xrightarrow{\Phi} & T^*S \\ & \searrow^{\pi} & \downarrow \\ & & S \end{array}$$

commutes; and for $s \in S$ let

$$\Phi_s : Z_s \to T_s^* S$$

be the restriction of Φ to Z_s.

Definition 4 *Φ is a* **moment map** *if, for all s and all $z \in Z_s$,*

$$(d\Phi_s)_z : T_z Z_s \to T_s^* S \tag{4.29}$$

is the transpose of (4.28).

Note that this condition determines Φ_s up to an additive constant $\nu_s \in T_s^* S$ and hence, as in § 4.10.2, determines Φ up to a section, $s \to \nu_s$, of T^*S.

When does a moment map exist? By (4.28) a vector, $v \in T_s S$, defines, for every point, $z \in Z_s$, an element of $T^* Z_s$ and hence defines a one-form on Z_s which we will show to be closed. We will say that G is *exact* if for all s and all $v \in T_s S$ this one-form is exact, and we will prove below that the exactness of G is a necessary and sufficient condition for the existence of Φ.

Given a moment map, Φ, one gets from it an imbedding

$$(G, \Phi) : Z \to M \times T^* S \tag{4.30}$$

and as in the previous section we can ask how close this comes to being a Lagrangian imbedding. We will prove

Theorem 19 *The pull-back by (4.30) of the symplectic form on $M \times T^* S$ is the pull-back by π of a closed two-form μ on S.*

The cohomology class of this two-form is an intrinsic invariant of G (doesn't depend on the choice of Φ) and as in the last section one can show that this is the only obstruction to making (4.30) a Lagrangian imbedding.

Theorem 20 *If $[\mu] = 0$ there exists a moment map, Φ, for which the imbedding (4.30) is Lagrangian.*

Conversely we will prove

Theorem 21 *Let Φ be a map of Z into $T^* S$ lifting the map, π, of Z into S. Then if the imbedding (4.30) is Lagrangian Φ is a moment map.*

4.10.4 Proofs.

Let us go back to the map (4.28). If we hold s fixed but let z vary over Z_s, we see that each $\xi \in T_sS$ gives rise to a one form on Z_s. To be explicit, let us choose a trivialization of our bundle around Z_s so we have an identification

$$H : Z_s \times U \to \pi^{-1}(U)$$

where U is a neighborhood of s in S. If $t \mapsto s(t)$ is any curve on S with $s(0) = s,\ s'(0) = \xi$ we get a curve of maps $h_{s(t)}$ of $Z_s \to M$ where

$$h_{s(t)} = g_{s(t)} \circ H.$$

We thus get a vector field v^ξ along the map h_s

$$v^\xi : Z_s \to TM, \qquad v^\xi(z) = \frac{d}{dt}h_{s(t)}(z)_{|t=0}.$$

Then the one form in question is

$$\tau^\xi = h_s^*(i(v^\xi)\omega).$$

A direct check shows that this one form is exactly the one form described above (and hence is independent of all the choices). We claim that

$$d\tau^\xi = 0. \tag{4.31}$$

Indeed, the general form of the Weil formula (9.8) and the fact that $d\omega = 0$ gives

$$\left(\frac{d}{dt}h^*_{s(t)}\omega\right)_{|t=0} = dh_s^* i(v^\xi)\omega$$

and the fact that Λ_s is Lagrangian for all s implies that the left hand side and hence the right hand side is zero. Let us now assume that G is *exact*, i.e. that for all s and ξ the one form τ^ξ is exact. So

$$\tau^\xi = d\phi^\xi$$

for some C^∞ function ϕ^ξ on Z_s. The function ϕ^ξ is uniquely determined up to an additive constant (if Z is connected) which we can fix (in various ways) so that it depends smoothly on s and linearly on ξ. For example, if we have a cross-section $c : S \to Z$ we can demand that $\phi(c(s))^\xi \equiv 0$ for all s and ξ. Alternatively, we can equip each fiber Z_s with a compactly supported density dz_s which depends smoothly on s and whose integral over Z_s is one for each s. We can then demand that that $\int_{Z_s} \phi^\xi dz_s = 0$ for all ξ and s.

Suppose that we have made such choice. Then for fixed $z \in Z_s$ the number $\phi^\xi(z)$ depends linearly on ξ. Hence we get a map

$$\Phi_0 : Z \to T^*S, \qquad \Phi_0(z) = \lambda \Leftrightarrow \lambda(\xi) = \phi^\xi(z). \quad (4.32)$$

We shall see below (Theorem 23) that Φ_0 is a moment map by computing its derivative at $z \in Z$ and checking that it is the transpose of (4.28).

If Z is connected, our choice determines ϕ^ξ up to an additive constant $\mu(s,\xi)$ which we can assume to be smooth in s and linear in ξ. Replacing ϕ^ξ by $\phi^\xi + \mu(s,\xi)$ has the effect of making the replacement

$$\Phi_0 \mapsto \Phi_0 + \mu \circ \pi$$

where $\mu : S \to T^*S$ is the one form $\langle \mu_s, \xi \rangle = \mu(s,\xi)$

Let ω_S denote the canonical two form on T^*S.

Theorem 22 *There exists a closed two form ρ on S such that*

$$G^*\omega - \Phi^*\omega_S = \pi^*\rho. \quad (4.33)$$

If $[\rho] = 0$ then there is a one form ν on S such that if we set

$$\Phi = \Phi_0 + \nu \circ \pi$$

then

$$G^*\omega - \Phi^*\omega_S = 0. \quad (4.34)$$

As a consequence, the map

$$\tilde{G} : Z \to M^- \times T^*S, \qquad z \mapsto (G(z), \Phi(z)) \quad (4.35)$$

is a Lagrangian embedding.

Proof. We first prove a local version of the theorem. Locally, we may assume that $Z = X \times S$. This means that we have an identification of Z_s with X for all s. By the Weinstein tubular neighborhood theorem we may assume (locally) that $M = T^*X$ and that for a fixed $s_0 \in S$ the Lagrangian submanifold Λ_{s_0} is the zero section of T^*X and that the map

$$G : X \times S \to T^*X$$

is given by

$$G(x,s) = d_X\psi(x,s)$$

where $\psi \in C^\infty(X \times S)$. In local coordinates $x_1, \dots, x_k$ on X, this reads as

$$G(x,s) = \frac{\partial \psi}{\partial x_1}dx_1 + \cdots + \frac{\partial \psi}{\partial x_k}dx_k.$$

In terms of these choices, the maps $h_{s(t)}$ used above are given by

$$h_{s(t)}(x) = d_X\psi(x,s(t))$$

and so (in local coordinates) on X and on S the vector field v^ξ is given by

$$v^\xi(z) = \frac{d}{dt}h_{s(t)}(z)_{|t=0} =$$

$$\frac{\partial^2\psi}{\partial x_1 \partial s_1}\xi_1\frac{\partial}{\partial p_1} + \cdots + \frac{\partial^2\psi}{\partial x_1 \partial s_r}\xi_r\frac{\partial}{\partial p_1} + \cdots + \frac{\partial^2\psi}{\partial x_k \partial s_r}\xi_r\frac{\partial}{\partial p_k}$$

where $r = \dim S$. We can write this more compactly as

$$\frac{\partial\langle d_S\psi, \xi\rangle}{\partial x_1}\frac{\partial}{\partial p_1} + \cdots + \frac{\partial\langle d_S\psi, \xi\rangle}{\partial x_k}\frac{\partial}{\partial p_k}.$$

Taking the interior product of this with $\sum dq_i \wedge dp_i$ gives

$$-\frac{\partial\langle d_S\psi, \xi\rangle}{\partial x_1}dq_1 - \cdots - \frac{\partial\langle d_S\psi, \xi\rangle}{\partial x_k}dq_k$$

and hence the one form τ^ξ is given by

$$-d_X\langle d_S\psi, \xi\rangle.$$

so we may choose

$$\Phi(x,s) = -d_S\psi(x,s).$$

Thus

$$G^*\alpha_X = d_X\psi, \qquad \Phi^*\alpha_S = -d_S\psi$$

and hence

$$G^*\omega_X - \Phi^*\omega_S = -dd\psi = 0.$$

This proves a local version of the theorem.

We now pass from the local to the global: By uniqueness, our global Φ_0 must agree with our local

Φ up to the replacement $\Phi \mapsto \Phi + \mu \circ \pi$. So we know that

$$G^*\omega - \Phi_0^*\omega_S = (\mu \circ \pi)^*\omega_S = \pi^*\mu^*\omega_S.$$

Here μ is a one form on S regarded as a map $S \to T^*S$. But

$$d\pi^*\mu^*\omega_S = \pi^*\mu^* d\omega_S = 0.$$

So we know that $G^*\omega - \Phi_0^*\omega_S$ is a closed two form which is locally and hence globally of the form $\pi^*\rho$ where $d\rho = 0$. This proves (4.33).

Now suppose that $[\rho] = 0$ so we can write $\rho = d\nu$ for some one form ν on S. Replacing Φ_0 by $\Phi_0 + \nu$ replaces ρ by $\rho + \nu^*\omega_S$. But

$$\nu^*\omega_S = -\nu^* d\alpha_S = -d\nu = -\rho. \qquad \square$$

Remark. If $[\rho] \neq 0$ then we can not succeed by modifying Φ. But we can modify the symplectic form on T^*S replacing ω_S by $\omega_S - \pi_S^*\rho$ where π_S denotes the projection $T^*S \to S$.

4.10.5 The derivative of Φ.

We continue the current notation. So we have the map

$$\Phi : Z \to T^*S.$$

Fix $s \in S$. The restriction of Φ to the fiber Z_s maps $Z_s \to T_s^*S$. since T_s^*S is a vector space, we may identify its tangent space at any point with T_s^*S itself. Hence for $z \in Z_s$ we may regard $d\Phi_z$ as a linear map from T_zZ to T_s^*S. So we write

$$d\Phi_z : T_zZ_s \to T_s^*S. \tag{4.36}$$

On the other hand, recall that using the identifications (4.26) and (4.27) we got a map

$$dG_z : T_sS \to T_m^*\Lambda, \quad m = G(z)$$

and hence composing with $d(g_s)_z^* : T_m^*\Lambda \to T_z^*Z_s$ a linear map

$$\chi_z := d(g_s)_z^* \circ dG_z : T_sS \to T_z^*Z. \tag{4.37}$$

Theorem 23 *The maps $d\Phi_z$ given by (4.36) and χ_z given by (4.37) are transposes of one another.*

Proof. Each $\xi \in T_sS$ gives rise to a one form τ^ξ on Z_s and by definition, the value of this one form at $z \in Z_s$ is exactly $\chi_z(\xi)$. The function ϕ^ξ was defined on $Z_s$so as to satisfy $d\phi^\xi = \tau^\xi$. In other words, for $v \in T_zZ$

$$\langle \chi_z(\xi), v\rangle = \langle d\Phi_z(v), \xi\rangle. \quad \Box$$

Corollary 24 *The kernel of χ_z is the annihilator of the image of the map (4.36). In particular z is a regular point of the map $\Phi : Z_s \to T_s^*S$ if the map χ_z is injective.*

Corollary 25 *The kernel of the map (4.36) is the annihilator of the image of χ_z.*

4.10.6 A converse.

The following is a converse to Theorem 22:

Theorem 26 *If $\Phi : Z \to T^*S$ is a lifting of the map $\pi : Z \to S$ to T^*S and (G, Φ) is a Lagrangian imbedding of*

$$Z \to M^- \times T^*S$$

then Φ is a moment map.

Proof. it suffices to prove this in the local model described above where $Z = X \times S$, $M = T^*X$ and $G(x, s) = d_X\psi(x, s)$. If $\Phi : X \times S \to T^*S$ is a lifting of the projection $X \times S \to X$, then (G, Φ) can be viewed as a section of $T^*(X \times S)$ i.e. as a one form β on $X \times S$. If (G, Φ) is a Lagrangian imbedding then β is closed. Moreover, he (1,0) component of β is $d_X\psi$ so $\beta - d\psi$ is a closed one form of type (0,1), and hence is of the form $\mu \circ \pi$ for some closed one form on S. this shows that

$$\Phi = d_S\psi + \pi^*\mu$$

and hence, as verfied above, is a moment map. $\Box$

4.10.7 Back to families of symplectomorphisms.

Let us now specialize to the case of a parametrized family of symplectomorphisms. So let (M, ω) be a symplectic manifold, S a manifold and

$$F : M \times S \to M$$

a smooth map such that

$$f_s : M \to M$$

is a symplectomophism for each s, where $f_s(m) = F(m, s)$. We can apply the results of the preceding section where now $\Lambda_s \subset M \times M^-$ is the graph of f_s (and the M of the preceding section is replaced by$M \times M^-$) and so

$$G : M \times S \to M \times M^-, \qquad G(m, s) = (m, F(m, s)). \tag{4.38}$$

Theorem 22 says that get a map

$$\Phi : M \times S \to T^*S$$

and a moment Lagrangian

$$\Gamma_\Phi \subset M \times M^- \times T^*S.$$

The equivariant situation.

Suppose that a compact Lie group K acts as fiber bundle automorphisms of $\pi : Z \to S$ and acts as symplectomorphisms of M. Suppose further that the fibers of Z are compact and equipped with a density along the fiber which is invariant under the group action. (For example, we can put any density on Z_s varying smoothly on s and then replace this density by the one obtained by averaging over the group.) Finally suppose that the map G is equivariant for the group actions of K on Z and on M. Then the map $\tilde{G}$ can be chosen to be equivariant for the actions of K on Z and the induced action of K on $M \times T^*S$.

More generally we want to consider situations where a Lie group K acts on Z as fiber bundle automorphisms and on M and where we know by explicit construction that the map $\tilde{G}$ can be chosen to be equivariant .

Hamiltonian group actions.

Let us specialize further by assuming that S is a Lie group K and that $F : M \times K \to M$ is a Hamiltonian group action. So we have a map

$$G : M \times K \to M \times M^-, \qquad (m, a) \mapsto (m, am).$$

Let K act on $Z = M \times K$ via its left action on K so $a \in K$ acts on Z as

$$a : (m, b) \mapsto (m, ab).$$

We expect to be able to construct $\tilde{G} : M \times K \to T^*K$ so as to be equivariant for the action of K on $Z = M \times K$ and the induced action of K on T^*K.

To say that the action is Hamiltonian with moment map $\Psi : M \to \mathfrak{k}^*$ is to say that

$$i(\xi_M)\omega = -d\langle \Psi, \xi \rangle.$$

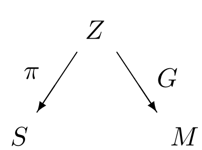
Thus under the left invariant identification of T^*K with $K \times \mathfrak{k}^*$ we see that Ψ determines a map

$$\Phi : M \times K \to T^*K, \qquad \Phi(m, a) = (a, \Psi(m)).$$

So our Φ of (4.32) is indeed a generalization of the moment map for Hamiltonian group actions.

4.11 Double fibrations.

The set-up described in § 4.10.2 has some legitimate applications of its own. For instance suppose that the diagram



is a double fibration: i.e., both π and G are fiber mappings and the map

$$(G, \pi) : Z \to M \times S$$

is an imbedding. In addition, suppose there exists a moment map $\Phi : Z \to T^*S$ such that

$$(G, \Phi) : Z \to M \times T^*S \tag{4.39}$$

is a Lagrangian imbedding. We will prove

Theorem 27 *The moment map $\Phi : Z \to T^*S$ is a co-isotropic immersion.*

Proof We leave as an exercise the following linear algebra result.

Lemma 1 *Let V and W be symplectic vector spaces and Γ a Lagrangian subspace of $V \times W$. Suppose the projection of Λ into V is surjective. Then the projection of Γ into W is injective and its image is a co-isotropic subspace of W.*

To prove the theorem let Γ_Φ be the image of the imbedding (4.39). Then the projection, $\Gamma_\phi \to M$, is just the map, G; so by assumption it is a submersion. Hence by the lemma, the projection, $\Gamma_\Phi \to T^*S$, which is just the map, Φ, is a co-isotropic immersion.

The most interesting case of the theorem above is the case when Φ is an imbedding. Then its image, Σ, is a co-isotropic submanifold of T^*S and M is just the quotient of Σ by its null-foliation. This description of M gives one, in principle, a method for quantizing M as a Hilbert subspace of $L_2(S)$. (For examples of how this method works in practice, see [**?**].)

4.11.1 The moment image of a family of symplectomorphisms

As in §4.10.7 let M be a symplectic manifold and let $\{f_s\,,\ s \in S\}$ be an exact family of symplectomorphisms. Let

$$\Phi : M \times S \to T^*S$$

be the moment map associated with this family and let

$$\Gamma = \{(m, f_s(m))\,,\ \Phi(m,s)\,;\ (m,s) \in M \times S\} \quad (4.40)$$

be its moment Lagrangian. From the perspective of §4.4, Γ is a morphism or “map”

$$\Gamma : M^- \times M \Rightarrow T^*S$$

mapping the categorical “points” (Lagrangian submanifolds) of $M^- \times M$ into the categorical “points” (Lagrangian submanifolds) of T^*S. Let Λ_Φ be the image with respect to this “map” of the diagonal, Δ, in $M \times M$. In more prosaic terms this image is just the image with respect to Φ (in the usual sense) of the subset

$$X = \{(m,s) \in M \times S\,;\ f_s(m) = m\} \quad (4.41)$$

of $M \times S$. As we explained in §4.2 this image will be a Lagrangian submanifold of T^*S only if one imposes transversal or clean intersection hypotheses on Γ and Δ. More explicitly let

$$\rho : \Gamma \to M \times M \tag{4.42}$$

be the projection of Γ into $M \times M$. The the pre-image in Γ of Δ can be identified with the set (4.41), and if ρ intersects Δ cleanly, the set (4.41) is a submanifold of $M \times S$ and we know from Theorem 7 that:

Theorem 28 *The composition,*

$$\Phi \circ j : X \to T^*S\,, \tag{4.43}$$

of Φ with the inclusion map, j, of X into $M \times S$ is a mapping of constant rank and its image, Δ_Φ, is an immersed Lagrangian submanifold of T^*S.

Remarks.

1. If the projection (4.42) intersects Δ transversally one gets a stronger result, Namely in this case the map (4.43) is a Lagrangian immersion.

2. If the map (4.43) is proper and its level sets are simply connected, then Λ_Φ is an imbedded Lagrangian submanifold of T^*S, and (4.43) is a fiber bundle mapping with X as fiber and Λ_Φ as base.

Let's now describe what this "moment image", Λ_Φ, of the moment Lagrangian look like in some examples:

4.11.2 The character Lagrangian.

Let K be the standard n-dimensional torus and $\mathfrak{k}$ its Lie algebra. Given a Hamiltonian action, τ, of K on a compact symplectic manifold, M, one has its usual moment mapping, $\phi : M \to \mathfrak{k}^*$; and if K acts faithfully the image of ϕ is a convex n-dimensional polytope, $\mathbf{P}_\Phi$.

If we consider the moment map $\Phi : M \to T^*K = K \times \mathfrak{k}^*$ in the sense of §4.10.2, The image of Φ in the categorical sense can be viewed as a labeled polytope in which the open $(n-k)$-dimensional faces of $\mathbf{P}_\Phi$ are labeled by k-dimensional subgroups of K. More explicitly, since M is compact, there are a finite number of subgroups of K occurring as stabilizer groups of points. Let

$$K_\alpha\,, \quad \alpha = 1, \dots, N \tag{4.44}$$

be a list of these subgroups and for each α let

$$M_{i,\alpha}\,, \quad i = 1, \dots, k_\alpha \tag{4.45}$$

be the connected components of the set of points whose stabilizer group is K_α. Then the sets

$$\phi(M_{i,\alpha}) = \mathbf{P}_{i,\alpha} \tag{4.46}$$

in $\mathfrak{k}^*$ are the open faces of $\mathbf{P}$ and the categorical image, Λ_Φ, of the set of symplectomorphisms $\{\tau_a\,,\, a \in K\}$ is the disjoint union of the Lagrangian manifolds

$$\Lambda_{i,\alpha} = K_\alpha \times \mathbf{P}_{i,\alpha} \tag{4.47}$$

4.11.3 The period–energy relation.

If one replaces the group, $K = \mathbb{T}^n$ in this example by the non-compact group, $K = \mathbb{R}^n$ one can't expect Λ_Φ to have this kind of polyhedral structure; however, Λ_Φ does have some interesting properties from the dynamical systems perspective. If $H : M \to (\mathbb{R}^n)^*$ is the moment map associated with the action of $\mathbb{R}^n$ onto M, the coordinates, H_i, of H can be viewed as Poisson–commuting Hamiltonians, and the $\mathbb{R}^n$ action is generated by their Hamiltonian vector fields, ν_{H_i}, i.e., by the map

$$s \in \mathbb{R}^n \to f_s = (\exp s_1 \nu_{H_1}) \dots (\exp s_n \nu_{H_n})\,. \tag{4.48}$$

Suppose now that $H : M \to (\mathbb{R}^n)^*$ is a proper submersion. Then each connected component, Λ, of Λ_Φ in $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$ is the graph of a map

$$H \to \left(\frac{\partial \psi}{\partial H_1}, \cdots, \frac{\partial \psi}{\partial H_n} \right)$$

over an open subset, U, of $(\mathbb{R}^n)^*$ with $\psi \in C^\infty(U)$, and, for $c \in U$, the element, $T = (T_1, \dots, T_n)$, $T_i = \frac{\partial \psi}{\partial H_i}(c)$, of $\mathbb{R}^n$ is the stabilizer of a connected component of periodic trajectories of the ν_{H_i}'s on the level set:

$$H_1 = c_1, \dots, H_n = c\,.$$

In particular all trajectories of ν_{H_i} have the same period, T_i, on this level set. This result is known in the theory of dynamical systems as the *period–energy relation.* In many examples of interest, the Legendre transform

$$\frac{\partial \psi}{\partial H} : U \to \mathbb{R}^n$$

is invertible, mapping U bijectively onto an open set, V, and in this case Λ is the graph of the "period mapping"

$$T \in V \to \frac{\partial \psi^*}{\partial T} \in (\mathbb{R}^n)^*$$

where ψ^* is the Legendre function dual to ψ.

4.11.4 The period–energy relation for families of symplectomorphisms.

We will show that something similar to this period–energy relation is true for families of symplectomorphisms providing we impose some rather strong assumptions on M and ω. Namely we will have to assume that ω is exact and that $H^1(M, \mathbb{R}) = 0$. Modulo these assumptions one can define, for a symplectomorphism, $f : M \to M$, and a fixed point, p of f, a natural notion of "the period of p".

The definition is the following. Choose a one-form, α, with $d\alpha = \omega$. Then

$$d(\alpha - f^*\alpha) = \omega - f^*\omega = 0$$

so

$$\alpha - f^*\alpha = d\psi \tag{4.49}$$

for some ψ in $C^\infty(M)$. (Unfortunately, ψ is only defined up to an additive constant, and one needs some "intrinsic" way of normalizing this constant. For instance, if ψ is bounded and M has finite volume one can require that the integral of ψ over M be zero, or if there is a natural base point, p_0, in M fixed by f, one can require that $\psi(p_0) = 0$.) Now, for every fixed point, p, set

$$T_p = \psi(p)\,. \tag{4.50}$$

This definition depends on the normalization we've made of the additive constant in the definition of ψ, but we claim that it's independent of the choice of α.

In fact, if we replace α by $\alpha + dg$, $g \in C^\infty(M)$, ψ gets changed to $\psi + f^*g - g$ and at the fixed point, p,

$$\psi(p) + (f^*g - g)(p) = \psi(p)\,,$$

so the definition (4.40) doesn't depend on α.

There is also a dynamical systems method of defining these periods. By a variant of the mapping torus construction of Smale one can construct a contact manifold, W, which is topologically identical with the usual mapping torus of f, and on this manifold a contact flow having the following three properties.

1. M sits inside W and is a global cross-section of this flow.

2. f is the "first return" map.

3. If $f(p) = p$ the periodic trajectory of the flow through p has T_p as period.

Moreover, this contact manifold is unique up to contact isomorphism. (For details see [?] or [?].) Let's apply these remarks to the set-up we are considering in this paper. As above let $F : M \times S \to M$ be a smooth mapping such that for every s the map $f_s : M \to M$, mapping m to $F(m, s)$, is a symplectomorphism. Let us assume that

$$H^1(M \times S, \mathbb{R}) = 0.$$

Let π be the projection of $M \times S$ onto M. Then if α is a one-form on M satisfying $d\alpha = \omega$ and α_S is the canonical one-form on T^*S the moment map $\Phi : M \times S \to M$ associated with F has the defining property

$$\pi^*\alpha - F^*\alpha + \Phi^*\alpha_S = d\psi \tag{4.51}$$

for some ψ in $C^\infty(M \times S)$. Let's now restrict both sides of (4.51) to $M \times \{s\}$. Since Φ maps $M \times \{s\}$ into T_s^*, and the restriction of α_S to T_s^* is zero we get:

$$\alpha - f_s^*\alpha = d\psi_s \tag{4.52}$$

where $\psi_s = \psi_{|M \times \{s\}}$.

Next let X be the set, (4.41), i.e., the set:

$$\{(m, s) \in M \times S\,, \quad F(m, s) = m\}$$

and let's restrict (4.51) to X. If j is the inclusion map of X into $M \times S$, then $F \circ j = \pi$; so

$$j^*(\pi^*\alpha - F^*\alpha) = 0$$

and we get from (4.51)

$$j^*(\phi^*\alpha_S - d\psi) = 0\,. \tag{4.53}$$

The identities, (4.52) and (4.53) can be viewed as a generalization of the period–energy relation. For instance, suppose the map

$$\tilde{F} : M \times S \to M \times M$$

mapping (m, s) to $(m\,,\, F(m, s))$ is transversal to Δ. Then by Theorem 28 the map $\Phi \circ j : X \to T^*S$ is a Lagrangian immersion whose image is Λ_Φ. Since $\tilde{F}$ intersects Δ transversally, the map

$$\tilde{f}_s : M \to M \times M\,, \quad \tilde{f}_s(m) = (m, f_s(m))\,,$$

intersects Δ transversally for almost all s, and if M is compact, f_s is Lefschetz and has a finite number of fixed points, $p_i(s)$, $i = 1, \ldots, k$. The functions, $\psi_i(s) = \psi(p_i(s), s)$, are, by (4.52), the periods of these fixed points and by (4.53) the Lagrangian manifolds

$$\Lambda_{\psi_i} = \{(s, \xi) \in T^*S \quad \xi = d\psi_i(s)\}$$

are the connected components of Λ_Φ.

4.12 The category of exact symplectic manifolds and exact canonical relations.

4.12.1 Exact symplectic manifolds.

Let (M, ω) be a symplectic manifold. It is possible that the symplectic form ω is exact, that is, that $\omega = -d\alpha$ for some one form α. When this happens, we say that (M, α) is an **exact sympletic manifold**. In other words, an exact symplectic manifold is a pair consisting of a manifold M together with a one form α such that $\omega = -d\alpha$ is of maximal rank. The main examples for us, of course, are cotangent bundles with their canonical one forms. Observe that

Proposition 10 *No positive dimensional compact symplectic manifold can be exact.*

Indeed, if (M, ω) is a symplectic manifold with M compact, then

$$\int_M \omega^d > 0$$

where $2d = \dim M$ assuming that $d > 0$. But if $\omega = -d\alpha$ then

$$\omega^d = -d\left(\alpha \wedge \omega^{d-1}\right)$$

and so $\int_M \omega^d = 0$ by Stokes' theorem. □

4.12.2 Exact Lagrangian submanifolds of an exact symplectic manifold.

Let (M, α) be an exact symplectic manifold and Λ a Lagrangian submanifold of (M, ω) where $\omega = -d\alpha$. Let

$$\beta_\Lambda := \iota_\Lambda^* \alpha \tag{4.54}$$

where

$$\iota_\Lambda : \Lambda \to M$$

is the embedding of Λ as a submanifold of M. So

$$d\beta_\Lambda = 0.$$

Suppose that β_Λ is exact, i.e. that $\beta_\Lambda = d\psi$ for some function ψ on Λ. (This will always be the case, for example, if Λ is simply connected.) We then call Λ an **exact** Lagrangian submanifold and ψ a choice of **phase function** for Λ.

Another important class of examples is where $\beta_\Lambda = 0$, in which case we can choose ψ to be locally constant. For instance, if $M = T^*X$ and $\Lambda = N^*(Y)$ is the conormal bundle to a submanifold $Y \subset X$ then we know that the restriction of α_X to $N^*(V)$ is 0.

4.12.3 The sub"category" of $\mathcal{S}$ whose objects are exact.

Consider the category whose objects are exact symplectic manifolds and whose morphisms are canonical relations between them. So let (M_1, α_1) and (M_2, α_2) be exact symplectic manifolds. Let

$$\mathrm{pr}_1 : M_1 \times M_2 \to M_1, \quad \mathrm{pr}_2 : M_1 \times M_2 \to M_2$$

be projections onto the first and second factors. Let

$$\alpha := -\operatorname{pr}_1^* \alpha_1 + \operatorname{pr}_2^* \alpha_2.$$

Then $-d\alpha$ gives the symplectic structure on $M_1^- \times M_2$.

To say that $\Gamma \in \operatorname{Morph}(M_1, M_2)$ is to say that Γ is a Lagrangian submanifold of $M_1^- \times M_2$. Let $\iota_\Gamma : \Gamma \to M_1^- \times M_2$ denote the inclusion map, and define, as above:

$$\beta_\Gamma := \iota_\Gamma^* \alpha.$$

We know that $d\beta_\Gamma = \iota^* d\alpha = 0$. So every canonical relation between cotangent bundles comes equipped with a closed one form.

Example: the canonical relation of a map.

Let $f : X_1 \to X_2$ be a smooth map and Γ_f the corresponding canonical relation from $M_1 = T^*X_1$ to $M_2 = T^*X_2$. By definition $\Gamma_f = (\varsigma_1 \times \mathrm{id}) N^*(\operatorname{graph}(f))$ and we know that the canonical one form vanishes on any conormal bundle. Hence

$$\beta_{\Gamma_f} = 0.$$

So if Γ is a canonical relation coming from a smooth map, its associated one form vanishes. We want to consider an intermediate class of Γ's - those whose associated one forms are exact.

Before doing so, we must study the behavior of the β_Γ under composition.

4.12.4 Functorial behavior of β_Γ.

Let (M_i, α_i) $i = 1, 2, 3$ be exact symplectic manifolds and

$$\Gamma_\in \operatorname{Morph}(M_1, M_2), \quad \Gamma_2 \in \operatorname{Morph}(M_2, M_3)$$

be cleanly composible canonical relations. Recall that we defined

$$\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$$

to consist of all (m_1, m_2, m_2, m_3) and we have the fibration

$$\kappa : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_2, \;\; \kappa(m_1, m_2, m_2, m_3) = (m_1, m_3).$$

We also have the projections

$$\varrho_1 : \Gamma_2 \star \Gamma_1 \to \Gamma_1, \quad \varrho_1((m_1, m_2, m_2, m_3)) = (m_1, m_2)$$

and

$$\varrho_2 : \Gamma_2 \star \Gamma_1 \to \Gamma_2, \quad \varrho((m_1, m_2, m_2, m_3)) = (m_2, m_3).$$

We claim that

$$\kappa^* \beta_{\Gamma_2 \circ \Gamma_1} = \varrho_1^* \beta_{\Gamma_1} + \varrho_2^* \beta_{\Gamma_2}. \tag{4.55}$$

Proof. Let π_1 and ρ_1 denote the projections of Γ_1 onto M_1 and M_2, and let π_2 and ρ_2 denote the projections of Γ_1 onto M_2 and M_3, so that

$$\rho_1 \varrho_1 = \pi_2 \varrho_2$$

both maps sending (m_1, m_2, m_2, m_3) to m_2. So

$$\beta_{\Gamma_1} = -\pi_1^* \alpha_1 + \rho_1^* \alpha_2 \quad \text{and} \quad \beta_{\Gamma_2} = -\pi_2^* \alpha_2 + \rho_2^* \alpha_3.$$

Thus

$$\varrho_1^* \beta_{\Gamma_1} + \varrho_2^* \beta_{\Gamma_2} = -\varrho_1^* \pi_1^* \alpha_1 + \varrho_2^* \rho_2^* \alpha_3 = \kappa^* \beta_{\Gamma_2 \circ \Gamma_1}. \quad \square$$

As a corollary we se that if $\beta_{\Gamma_i} = d\psi_i, \quad i = 1, 2$ then

$$\kappa^* \beta_{\Gamma_2 \circ \Gamma_1} = d \left(\varrho_1^* \psi_1 + \varrho_2^* \psi_2 \right).$$

So let us call a canonical relation **exact** if its associated (closed) one form is exact. We see that if we restrict ourselves to canonical relations which are exact, then we obtain a sub"category" of the "category" whose objects are exact symplectic manifolds and whose morphisms are exact canonical relations.

4.12.5 Defining the "category" of exact symplectic manifolds and canonical relations.

If Γ is an exact canonical relation so that $\beta_\Gamma = d\psi$, then ψ is only determined up to an additive constant (if Γ is connected). But we can *enhance* our sub"category" by specifying ψ. That is, we consider the "category" whose objects are exact symplectic manifolds and whose morphisms are pairs (Γ, ψ) where Γ is an exact canonical relation and $\beta_\Gamma = d\psi$.

Then composition is defined as follows: If Γ_1 and Γ_2 are cleanly composible, then we define

$$(\Gamma_2, \psi_2) \circ (\Gamma_1, \psi_1) = (\Gamma_2 \circ \Gamma_1, \psi) \tag{4.56}$$

where the (local) additive constant in ψ is determined by

$$\kappa^* \psi = \varrho_1^* \psi_1 + \varrho_2^* \psi_2. \tag{4.57}$$

We shall call this enhanced sub"category" the "category" of **exact canonical relations** (between cotangent bundles).

An important sub"category" of this "category" is where the objects are cotangent bundles with their canonical one forms.

4.12.6 Pushforward via a map in the "category" of exact canonical relations between cotangent bundles.

As an illustration of the composition law (4.56) consider the case where Λ_Z is an exact Lagrangian submanifold of T^*Z so that the restriction of the one form of T^*Z to Λ is given by $d\psi_\Lambda$. We consider Λ as an element of Morph(pt., T^*Z) so we can take (Λ, ψ) as the (Γ_1, ψ_1) in (4.56). Let $f : Z \to X$ be a smooth map and take Γ_2 in (4.56) to be Γ_f. We know that the one form associated to Γ_f vanishes. In our enhanced category we must specify the function whose differential vanishes on Γ_f - that is we must pick a (local) constant c. So in (4.56) we have $(\Gamma_2, \psi_2) = (\Gamma_f, c)$. Assume that the Γ_f and Λ are composible. Recall that then $\Gamma_f \circ \Lambda_Z = df_*(\Lambda_Z)$ consists of all (x, ξ) where $x = f(z)$ and $(z, df^*(\xi)) \in \Lambda$. Then (4.56) says that

$$\psi(x, \xi) = \psi_\Lambda(z, \eta) + c. \tag{4.58}$$

In the next chapter and in Chapter 8 will be particularly interested in the case where f is a fibration. So we are given a fibration $\pi : Z \to X$ and we take $\Lambda_Z = \Lambda_\phi$ to be a horizontal Lagrangian submanifold of T^*Z. We will also assume that the composition in (4.56) is transversal. In this case the pushforward map $d\pi_*$ gives a diffeomorphism of Λ_ϕ with $\Lambda := df_*(\Lambda_\phi)$. In our applications, we will be given the pair (Λ, ψ) and we will regard (4.58) as *fixing the* *arbitrary constant* in ϕ rather than in Γ_f whose constant we take to be 0.

Chapter 5

Generating functions.

In this chapter we continue the study of canonical relations between cotangent bundles. We begin by studying the canonical relation associated to a map in the special case when this map is a fibration. This will allow us to generalize the local description of a Lagrangian submanifold of T^*X that we studied in Chapter 1. In Chapter 1 we showed that a *horizontal* Lagrangian submanifold of T^*X is locally described as the set of all $d\phi(x)$ where $\phi \in C^\infty(X)$ and we called such a function a "generating function". The purpose of this chapter is to generalize this concept by introducing the notion of a generating function relative to a fibration.

5.1 Fibrations.

In this section we will study in more detail the canonical relation associated to a fibration. So let X and Z be manifolds and

$$\pi : Z \to X$$

a smooth fibration. Then

$$\Gamma_\pi \in \mathrm{Morph}(T^*Z, T^*X)$$

consists of all $(z, \xi, x, \eta) \in T^*Z \times T^*X$ such that

$$x = \pi(z) \quad \text{and} \quad \xi = (d\pi_z)^*\eta.$$

Then

$$\mathrm{pr}_1 : \Gamma_\pi \to T^*Z, \qquad (z,\xi,x,\eta) \mapsto (z,\xi)$$

maps Γ_π bijectively onto the sub-bundle of T^*Z consisting of those covectors which vanish on tangents to the fibers. We will call this sub-bundle the **horizontal sub-bundle** and denote it by H^*Z. So at each $z \in Z$, the fiber of the horizontal sub-bundle is

$$H^*(Z)_z = \{(d\pi_z)^*\eta,\ \eta \in T^*_{\pi(z)}X\}.$$

Let Λ_Z be a Lagrangian submanifold of T^*Z which we can also think of as an element of Morph(pt., T^*Z). We want to study the condition that Γ_π and Λ_Z be composible so that we be able to form

$$\Gamma_\pi(\Lambda_Z) = \Gamma_\pi \circ \Lambda_Z$$

which would then be a Lagrangian submanifold of T^*X. If $\iota : \Lambda_Z \to T^*Z$ denotes the inclusion map then the clean intersection part of the composibility condition requires that ι and pr_1 intersect cleanly. This is the same as saying that Λ_Z and H^*Z intersect cleanly in which case the intersection

$$F := \Lambda_Z \cap H^*Z$$

is a smooth manifold and we get a smooth map $\kappa : F \to T^*X$. The remaining hypotheses of Theorem 9 require that this map be proper and have connected and simply connected fibers.

A more restrictive condition is that intersection be transversal, i.e. that

$$\Lambda_Z \pitchfork H^*Z$$

in which case we always get a Lagrangian immersion

$$F \to T^*X, \quad (z, d\pi_z^*\eta) \mapsto (\pi(z),\eta).$$

The additional composibility condition is that this be an embedding.

Let us specialize further to the case where Λ_Z is a horizontal Lagrangian submanifold of T^*Z. That is, we assume that

$$\Lambda_Z = \Lambda_\phi = \gamma_\phi(Z) = \{(z, d\phi(z))\}$$

as in Chapter 1. When is

$$\Lambda_\phi \pitchfork H^*Z?$$

Now H^*Z is a sub-bundle of T^*Z so we have the exact sequence of vector bundles

$$0 \to H^*Z \to T^*Z \to V^*Z \to 0 \tag{5.1}$$

where

$$(V^*Z)_z = T_z^*Z/(H^*Z)_z = T_z^*(\pi^{-1}(x)), \quad x = \pi(z)$$

is the cotangent space to the fiber through z.

Any section $d\phi$ of T^*Z gives a section $d_{vert}\phi$ of V^*Z by the above exact sequence, and $\Lambda_\phi \pitchfork H^*Z$ if and only if this section intersects the zero section of V^*Z transversally. If this happens,

$$C_\phi := \{z \in Z | (d_{vert}\phi)_z = 0\}$$

is a submanifold of Z whose dimension is $\dim X$. Furthermore, at any $z \in C_\phi$

$$d\phi_z = (d\pi_z)^*\eta \quad \text{for a unique} \quad \eta \in T^*_{\pi(z)}X.$$

Thus Λ_ϕ and Γ_π are transversally composible if and only if

$$C_\phi \to T^*X, \quad z \mapsto (\pi(z), \eta)$$

is a Lagrangian embedding in which case its image is a Lagrangian submanifold

$$\Lambda = \Gamma_\pi(\Lambda_\phi) = \Gamma_\pi \circ \Lambda_\phi$$

of T^*X. When this happens we say that ϕ is a **a transverse generating function of Λ with respect to the fibration** (Z, π).

If Λ_ϕ and Γ_π are merely cleanly composible, we say that ϕ is a **clean** generating function with respect to π.

If ϕ is a transverse generating function for Λ with respect to the fibration, π, and $\pi_1 : Z_1 \to Z$ is a fibration over Z, then it is easy to see that $\phi_1 = \pi_1^*\phi$ is a clean generating function for Λ with respect to the fibration, $\pi \circ \pi_1$; and we will show in the next section that there is a converse result: Locally *every* clean generating function can be obtained in this way from a transverse generating function. For this reason it will suffice, for most of the things we'll be doing in this chapter, to work with transverse generating functions; and to simplify notation, we will henceforth, unless otherwise stated, use the terms "generating function" and "transverse generating function" interchangeably.

5.1.1 Transverse vs. clean generating functions.

Locally we can assume that Z is the product, $X \times S$, of X with an open subset, S, of $\mathbb{R}^k$ with standard coordinates $s_1, \dots, s_k$. Then H^*Z is defined by the equations, $\eta_1 = \cdots = \eta_k = 0$, where the η_i's are the standard cotangent coordinates on T^*S; so $\Lambda_\phi \cap H^*Z$ is defined by the equations

$$\frac{\partial \phi}{\partial s_i} = 0\,, \quad i = 1, \dots, k\,.$$

Let C_ϕ be the subset of $X \times S$ defined by these equations. Then if Λ_ϕ intersects H^*Z cleanly, C_ϕ is a submanifold of $X \times S$ of codimension $r \leq k$; and, at every point $(x_0, s_0) \in C_\phi$, C_ϕ can be defined locally near (x_0, s_0) by r of these equations, i.e., modulo repagination, by the equations

$$\frac{\partial \phi}{\partial s_i} = 0\,, \quad i = 1, \dots, r\,.$$

Moreover these equations have to be independent: the tangent space at (x_0, s_0) to C_ϕ has to be defined by the equations

$$d\left(\frac{\partial \phi}{\partial s_i}\right)_{(x_0, \xi_0)} = 0\,, \quad i = 1, \dots, r\,.$$

Suppose $r < k$ (i.e., suppose this clean intersection is not transverse). Since $\partial \phi / \partial s_k$ vanishes on C_ϕ, there exist C^∞ functions, $g_i \in C^\infty(X \times S)$, $i = 1, \dots, r$ such that

$$\frac{\partial \phi}{\partial s_k} = \sum_{i=1}^{r} g_i \frac{\partial \phi}{\partial s_i}\,.$$

In other words, if ν is the vertical vector field

$$\nu = \frac{\partial}{\partial s_k} - \sum_{i=1}^{r} g_i(x, s) \frac{\partial}{\partial s_i}$$

then $D_\nu \phi = 0$. Therefore if we make a change of vertical coordinates

$$(s_i)_{\text{new}} = (s_i)_{\text{new}}(x, s)$$

so that in these new coordinates

$$\nu = \frac{\partial}{\partial s_k}$$

this equation reduces to

$$\frac{\partial}{\partial s_k}\phi(x, s) = 0\,,$$

so, in these new coordinates,

$$\phi(x, s) = \phi(x, s_1, \dots, s_{k-1})\,.$$

Iterating this argument we can reduce the number of vertical coordinates so that $k = r$, i.e., so that ϕ is a transverse generating function in these new coordinates. In other words, a clean generating function is just a transverse generating function to which a certain number of vertical "ghost variables" ("ghost" meaning that the function doesn't depend on these variables) have been added. The number of these ghost variables is called the *excess* of the generating function. (Thus for the generating function in the paragraph above, its excess is $k - r$.) More intrinsically the *excess is the difference between the dimension of the critical set C_ϕ of ϕ and the dimension of X.*

As mentioned above, unless specified otherwise, we assume that our generating function are transverse generating functions.

5.2 The generating function in local coordinates.

Suppose that X is an open subset of $\mathbb{R}^n$, that

$$Z = X \times \mathbb{R}^k$$

that π is projection onto the first factor, and that (x, s) are coordinates on Z so that $\phi = \phi(x, s)$. Then $C_\phi \subset Z$ is defined by the k equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \dots, k.$$

and the transversality condition is that these equations be functionally independent. This amounts to the hypothesis that their differentials

$$d\left(\frac{\partial \phi}{\partial s_i}\right) \qquad i = 1, \dots k$$

be linearly independent. Then $\Lambda \subset T^*X$ is the image of the embedding

$$C_\phi \to T^*X, \quad (x, s) \mapsto \frac{\partial \phi}{\partial x} = d_X \phi(x, s).$$

5.3 Example - a generating function for a conormal bundle.

Suppose that

$$Y \subset X$$

is a submanifold defined by the k functionally independent equations

$$f_1(x) = \cdots = f_k(x) = 0.$$

Let $\phi : X \times \mathbb{R}^k \to \mathbb{R}$ be the function

$$\phi(x, s) := \sum_i f_i(x) s_i. \tag{5.2}$$

We claim that

$$\Lambda = \Gamma_\pi \circ \Lambda_\phi = N^*Y, \tag{5.3}$$

the conormal bundle of Y. Indeed,

$$\frac{\partial \phi}{\partial s_i} = f_i$$

so

$$C_\phi = Y \times \mathbb{R}^k$$

and the map

$$C_\phi \to T^*X$$

is given by

$$(x, s) \mapsto \sum s_i d_X f_i(x).$$

The differentials $d_X f_x$ span the conormal bundle to Y at each $x \in Y$ proving (5.3).

As a special case of this example, suppose that

$$X = \mathbb{R}^n \times \mathbb{R}^n$$

and that Y is the diagonal

$$\mathrm{diag}(X) = \{(x, x)\} \subset X$$

which may be described as the set of all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$x_i - y_i = 0, \quad i = 1, \ldots, n.$$

We may then choose

$$\phi(x, y, s) = \sum_i (x_i - y_i) s_i. \tag{5.4}$$

Now $\mathrm{diag}(X)$ is just the graph of the identity transformation so by Section 4.7 we know that $(\varsigma_1 \times \mathrm{id})(N^*(\mathrm{diag}(X))$ is the canonical relation giving the identity map on T^*X. By abuse of language we can speak of ϕ as the generating function of the identity canonical relation. (But we must remember the ς_1.)

5.4 Example. The generating function of a geodesic flow.

A special case of our generating functions with respect to a fibration is when the fibration is trivial, i.e. π is a diffeomorphism. Then the vertical bundle is trivial and we have no "auxiliary variables". Such a generating function is just a generating function in the sense of Chapter 1. For example, let X be a Riemannian manifold and let $\phi_t \in C^\infty(X \times X)$ be defined by

$$\phi_t(x, y) := \frac{1}{2t} d(x, y)^2, \tag{5.5}$$

where

$$t \neq 0.$$

Let us compute Λ_ϕ and $(\varsigma_1 \times \mathrm{id})(\Lambda_\phi)$. We first do this computation under the assumption that $X = \mathbb{R}^n$ and the metric occurring in (5.5) is the Euclidean metric so that

$$\begin{aligned}
\phi(x,y,t) &= \frac{1}{2t}\sum_i (x_i - y_i)^2 \\
\frac{\partial \phi}{\partial x_i} &= \frac{1}{t}(x_i - y_i) \\
\frac{\partial \phi}{\partial y_i} &= \frac{1}{t}(y_i - x_i) \quad \text{so} \\
\Lambda_\phi &= \{(x, \frac{1}{t}(x-y), y, \frac{1}{t}(y-x)\} \text{ and} \\
(\varsigma_1 \times \mathrm{id})(\Lambda_\phi) &= \{(x, \frac{1}{t}(y-x), y, \frac{1}{t}(y-x)\}.
\end{aligned}$$

In this last equation let us set $y - x = t\xi$, i.e.

$$\xi = \frac{1}{t}(y-x)$$

which is possible since $t \neq 0$. Then

$$(\varsigma_1 \times \mathrm{id})(\Lambda_\phi) = \{(x, \xi, x + t\xi, \xi)\}$$

which is the graph of the symplectic map

$$(x,\xi) \mapsto (x + t\xi, \xi).$$

If we identify cotangent vectors with tangent vectors (using the Eulidean metric) then $x + t\xi$ is the point along the line passing through x with tangent vector ξ a distance $t\|\xi\|$ out. The one parameter family of maps $(x,\xi) \mapsto (x+t\xi,\xi)$ is known as the geodesic flow. In the case of Euclidean space, the time t value of this flow is a diffeomorphism of T^*X with itself for every t. So long as $t \neq 0$ it has the generating function given by (5.5) with no need of auxiliary variables. When $t = 0$ the map is the identity and we need to introduce a fibration.

More generally, this same computation works on any geodesically convex Riemannian manifold:

A Riemannian manifold X is called **geodesically convex** if, given any two points x and y in X, there is a unique geodesic which joins them. We will show that the above computation of the generating function works for any geodesically convex Riemannian manifold. In fact, we will prove a more general result. Recall that geodesics on a Riemannian manifold can be described as follows: A Riemann metric on a manifold X is the same as a scalar product on each tangent space T_xX which varies smoothly with X. This induces an identification of TX with T^*X an hence a scalar product $\langle\ ,\ \rangle_x$ on each T^*X. This in turn induces the "kinetic energy" Hamiltonian

$$H(x,\xi) := \frac{1}{2}\langle \xi, \xi\rangle_x.$$

The principle of least action says that the solution curves of the corresponding vector field v_H project under $\pi : T^*X \to X$ to geodesics of X and every geodesic is the projection of such a trajectory.

An important property of the kinetic energy Hamilonian is that it is quadratic of degree two in the fiber variables. We will prove a theorem (see Theorem 29 below) which generalizes the above computation and is valid for any Hamiltonian which is homogeneous of degree $k \neq 1$ in the fiber variables and which satisfies a condition analogous to the geodesic convexity theorem. We first recall some facts about homogeneous functions and Euler's theorem.

Consider the one parameter group of dilatations $t \mapsto \mathfrak{d}(t)$ on any cotangent bundle T^*X:

$$\mathfrak{d}(t) : T^*X \to T^*X : \qquad (x,\xi) \mapsto (x, e^t\xi).$$

A function f is homogenous of degree k in the fiber variables if and only if

$$\mathfrak{d}(t)^*f = e^{kt}f.$$

For example, the principal symbol of a k-th order linear partial differential operator on X is a function on T^*X with which is a polynomial in the fiber variables and is homogenous of degree k.

Let $\mathcal{E}$ denote the vector field which is the infinitesimal generator of the one parameter group of dilatations. It is called the **Euler vector field**. Euler's theorem (which is a direct computation from the preceding equation) says that f is homogenous of degree k if and only if

$$\mathcal{E}f = kf.$$

Let $\alpha = \alpha_X$ be the canonical one form on T^*X. From its very definition (1.8) it follows that

$$\mathfrak{d}(t)^*\alpha = e^t\alpha$$

and hence that

$$D_{\mathcal{E}}\alpha = \alpha.$$

Since $\mathcal{E}$ is everywhere tangent to the fiber, it also follows from (1.8) that

$$i(\mathcal{E})\alpha = 0$$

and hence that

$$\alpha = D_{\mathcal{E}}\alpha = i(\mathcal{E})d\alpha = -i(\mathcal{E})\omega$$

where $\omega = \omega_X = -d\alpha$.

Now let H be a function on T^*X which is homogeneous of degree k in the fiber variables. Then

$$\begin{aligned}
kH = \mathcal{E}H &= i(\mathcal{E})dH \\
&= i(\mathcal{E})i(v_H)\omega \\
&= -i(v_H)i(\mathcal{E})\omega \\
&= i(v_H)\alpha \quad \text{and} \\
(\exp v_H)^*\alpha - \alpha &= \int_0^1 \frac{d}{dt}(\exp tv_H)^*\alpha dt \quad \text{with} \\
\frac{d}{dt}(\exp tv_H)^*\alpha &= (\exp tv_H)^*\left(i(v_H)d\alpha + di(v_H)\alpha\right) \\
&= (\exp tv_H)^*\left(-i(v_H)\omega + di(v_H)\alpha\right) \\
&= (\exp tv_H)^*\left(-dH + kdH\right) \\
&= (k-1)(\exp tv_H)^*dH \\
&= (k-1)d(\exp tv_H)^*H \\
&= (k-1)dH
\end{aligned}$$

since H is constant along the trajectories of v_H. So

$$(\exp v_H)^*\alpha - \alpha = (k-1)dH. \tag{5.6}$$

Remark. In the above calculation we assumed that H was smooth on all of T^*X including the zero section, effectively implying that H is a polynomial in the fiber variables. But the same argument will go through (if $k > 0$) if all we assume is that H (and hence v_H) are defined on $T^*X\setminus$ the zero section, in which case H can be a more general homogeneous function on $T^*X\setminus$ the zero section.

Now $\exp v_H \ : T^*X \to T^*X$ is symplectic map. Let

$$\Gamma := \text{graph } (\exp v_H),$$

so $\Gamma \subset T^*X^- \times T^*X$ is a Lagrangian submanifold. Suppose that the projection $\pi_{X\times X}$ of Γ onto $X \times X$ is a diffeomorphism, i.e. suppose that Γ is horizontal. This says precisely that for every $(x, y) \in X\times X$ there is a unique $\xi \in T^*_x X$ such that

$$\pi \exp v_H(x, \xi) = y.$$

In the case of the geodesic flow, this is guaranteed by the condition of geodesic convexity.

Since Γ is horizontal, it has a generating function ϕ such that

$$d\phi = \mathrm{pr}_2^* \alpha - \mathrm{pr}_1^* \alpha$$

where pr_i, $i = 1, 2$ are the projections of $T^*(X\times X) = T^*X \times T^*X$ onto the first and second factors. On the other hand pr_1 is a diffeomorphism of Γ onto T^*X. So

$$\mathrm{pr}_1 \circ (\pi_{X\times X|\Gamma})^{-1}$$

is a diffeomorphism of $X \times X$ with T^*X.

Theorem 29 *Assume the above hypotheses. Then up to an additive constant we have*

$$\left(\mathrm{pr}_1 \circ (\pi_{X\times X|\Gamma})^{-1}\right)^* [(k-1)H] = \phi.$$

In the case where $H = \frac{1}{2}\|\xi\|^2$ is the kinetic energy of a geodesically convex Riemann manifold, this says that

$$\phi = \frac{1}{2} d(x, y)^2.$$

Indeed, this follows immediately from(5.6). An immediate corollary (by rescaling) is that (5.5) is the generating function for the time t flow on a geodesically convex Riemannian manifold.

As mentioned in the above remark, the same theorem will hold if H is only defined on $T^*X \setminus \{0\}$ and the same hypotheses hold with $X \times X$ replaced by $X \times X \setminus \Delta$.

5.5 The generating function for the transpose.

Let

$$\Gamma \in \mathrm{Morph}(T^*X, T^*Y)$$

be a canonical relation, let

$$\pi : Z \to X \times Y$$

be a fibration and ϕ a generating function for Γ relative to this fibration. In local coordinates this says that $Z = X \times Y \times S$, that

$$C_\phi = \{(x, y, s) | \frac{\partial \phi}{\partial s} = 0\},$$

and that Γ is the image of C_ϕ under the map

$$(x, y, s) \mapsto (-d_X \phi, d_Y \phi).$$

Recall that

$$\Gamma^\dagger \in \text{Morph}(T^*Y, T^*X)$$

is given by the set of all (γ_2, γ_1) such that $(\gamma_1, \gamma_2) \in \Gamma$. So if

$$\kappa : X \times Y \to Y \times X$$

denotes the transposition

$$\kappa(x, y) = (y, x)$$

then

$$\kappa \circ \pi : Z \to Y \times X$$

is a fibration and $-\phi$ is a generating function for $\Gamma^\dagger$ relative to $\kappa \circ \pi$. Put more succinctly, if $\phi(x, y, s)$ is a generating function for Γ then

$$\psi(y, x, s) = -\phi(x, y, s) \text{ is a generating function for } \Gamma^\dagger. \tag{5.7}$$

For example, if Γ is the graph of a symplectomorphism, then $\Gamma^\dagger$ is the graph of the inverse diffeomorphism. So (5.7) says that $-\phi(y, x, s)$ generates the inverse of the symplectomorphism generated by $\phi(x, y, s)$.

This suggests that there should be a simple formula which gives a generating function for the composition of two canonical relations in terms of the generating function of each. This was one of Hamilton's great achievements - that, in a suitable sense to be described in the next section - the generating function for the composition is the sum of the individual generating functions.

5.6 The generating function for a transverse composition.

Let X_1, X_2 and X_3 be manifolds and

$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2), \quad \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$

be canonical relations which are transversally composible. So we are assuming in particular that the maps

$$\Gamma_1 \to T^*X_2, \quad (p_1, p_2) \mapsto p_2 \ \text{ and } \ \Gamma_2 \to T^*X_2, \ (q_2, q_3) \mapsto q_2$$

are transverse.

Suppose that

$$\pi_1 : Z_1 \to X_1 \times X_2, \quad \pi_2 : Z_2 \to X_2 \times X_3$$

are fibrations and that $\phi_i \in C^\infty(Z_i), \ i = 1, 2$ are generating functions for Γ_i with respect to π_i.

From π_1 and π_2 we get a map

$$\pi_1 \times \pi_2 : Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3.$$

Let

$$\Delta_2 \subset X_2 \times X_2$$

be the diagonal and let

$$Z := (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_2 \times X_3).$$

Finally, let

$$\pi : Z \to X_1 \times X_3$$

be the fibration

$$Z \to Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3 \to X_1 \times X_3$$

where the first map is the inclusion map and the last map is projection onto the first and last components. Let

$$\phi : Z \to \mathbb{R}$$

be the restriction to Z of the function

$$(z_1, z_2) \mapsto \phi_1(z_1) + \phi_2(z_2). \tag{5.8}$$

Theorem 30 *ϕ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration $\pi : Z \to X_1 \times X_3$.*

Proof. We may check this in local coordinates where the fibrations are trivial to that

$$Z_1 = X_1 \times X_2 \times S, \quad Z_2 = X_2 \times X_3 \times T$$

so

$$Z = X_1 \times X_3 \times (X_2 \times S \times T)$$

and π is the projection of Z onto $X_1 \times X_3$. Notice that X_2 has now become a factor in the *parameter space*. The function ϕ is given by

$$\phi(x_1, x_3, x_2, s, t) = \phi_1(x_1, x_2, s) + \phi_2(x_2, x_3, t).$$

For $z = (x_1, x_3, x_2, s, t)$ to belong to C_ϕ the following three conditions must be satisfied and be functionally independent:

- $\frac{\partial \phi_1}{\partial s}(x_1, x_2, s) = 0$, i.e. $z_1 = (x_1, x_2, s) \in C_{\phi_1}$.
- $\frac{\partial \phi_2}{\partial t} = 0$, i.e. $z_2 = (x_2, x_3, t) \in C_{\phi_2}$ and
- $$\frac{\partial \phi_1}{\partial x_2}(x_1, x_2, s) + \frac{\partial \phi_2}{\partial x_2}(x_2, x_3, t) = 0.$$

To show that these equations are functionally independent, we will rewrite them as the following system of equations on $X_1 \times X_3 \times X_2 \times X_2 \times S \times T$:

1. $\frac{\partial \phi_1}{\partial s}(x_1, x_2, s) = 0$, i.e. $z_1 = (x_1, x_2, s) \in C_{\phi_1}$,
2. $\frac{\partial \phi_2}{\partial t}(y_2, x_3, t) = 0$, i.e. $z_2 = (y_2, x_3, t) \in C_{\phi_2}$,
3. $x_2 = y_2$ and
4. $$\frac{\partial \phi_1}{\partial x_2}(x_1, x_2, s) + \frac{\partial \phi_2}{\partial x_2}(y_2, x_3, t) = 0.$$

It is clear that 1) and 2) are independent, and define the product $C_{\phi_1} \times C_{\phi_2}$ as a submanifold of $X_1 \times X_2 \times X_2 \times X_3$. So to show that 1)-4) are independent, we must show that 3) and 4) are an independent system of equations on $C_{\phi_1} \times C_{\phi_2}$.

From the fact that ϕ_1 is a generating function for Γ_1, we know that the map

$$\gamma_1 : C_{\phi_1} \to \Gamma_1, \quad \gamma_1(p_1) = \left(x_1, -\frac{\partial \phi_1}{\partial x_1}(p_1), x_2, \frac{\partial \phi_1}{\partial x_2}(p_1)\right)$$

where

$$(x_1, x_2) = \pi_1(p_1)$$

is a diffeomorphism. Similarly, the map

$$\gamma_2 : C_{\phi_1} \to \Gamma_2, \quad \gamma_2(p_2) = \left(x_2, -\frac{\partial \phi_2}{\partial x_2}(p_2), x_3, \frac{\partial \phi_2}{\partial x_3}(p_2)\right)$$

where

$$(x_2, x_3) = \pi_2(p_2)$$

is a diffeomorphism.

So if we set $M_i := T^*X_i,\ i = 1,2,3$ we can write the preceding diffeomorphisms as

$$\gamma_i(p_i) = (m_i, m_{i+1}), i = 1, 2$$

where

$$m_i = (x_i, -\frac{\partial \phi_i}{\partial x_i}(p_i)), \qquad m_{i+1} = (x_{i+1}, \frac{\partial \phi_i}{\partial x_{i+1}}(p_i)) \tag{5.9}$$

and the x_i are as above. We have the diffeomorphism

$$\gamma_1 \times \gamma_2 : \ C_{\phi_1} \times C_{\phi_2} \to \Gamma_1 \times \Gamma_2$$

and the map

$$\kappa : \Gamma_1 \times \Gamma_2 \to M_2 \times M_2, \ \ \kappa(m_1, m_2, n_2, m_3) = (m_2, n_2).$$

This map κ is assumed to be transverse to the diagonal Δ_{M_2}, and hence the map

$$\lambda : C_{\phi_1} \times C_{\phi_2} \to M_2 \times M_2, \quad \lambda := \kappa \circ (\gamma_1 \times \gamma_2)$$

is transverse to Δ_{M_2}. This transversality is precisely the functional independence of conditions 3) and 4) above.

The manifold $\Gamma_2 \star \Gamma_1$ was defined to be $\kappa^{-1}(\Delta_{M_2})$ and the second condition for transverse composibility was that the map

$$\rho : \Gamma_2 \star \Gamma_1 \to M_1^- \times M_3, \ \ \rho(m_1.m_2, m_2, m_3) = (m_1, m_3)$$

be an embedding whose image is then defined to be $\Gamma_2 \circ \Gamma_1$. The diffeomorphism $\gamma_1 \times \gamma_2$ then shows that the critical set C_ϕ is mapped diffeomorphically onto $\Gamma_2 \star \Gamma_1$. Here ϕ is defined by (5.8). Call this diffeomorphism τ. So

$$\tau : C_\phi \cong \Gamma_2 \star \Gamma_1.$$

Thus

$$\rho \circ \tau : \; C_\phi \to \Gamma_2 \circ \Gamma_1$$

is a diffeomorphism, and (5.9) shows that this diffeomorphism is precisely the one that makes ϕ a generating function for $\Gamma_2 \circ \Gamma_1$. □

In the next section we will show that the arguments given above apply, essentially without change, to *clean* generating functions, since, as we saw in Section 5.1.1, clean generating functions are just transverse generating functions to which a number of vertical "ghost variables" have been added.

5.7 Generating functions for clean composition of canonical relations between cotangent bundles.

Suppose that the canonical relation, Γ_1 and Γ_2 are *cleanly* composible. Let $\phi_1 \in C^\infty(X_1 \times X_2 \times S)$ and $\phi_2 \in C^\infty(X_2 \times X_3 \times T)$ be *transverse* generating functions for Γ_1 and Γ_2 and as above let

$$\phi(x_1, x_3, x_2, s, t) = \phi_1(x_1, x_2, s) + \phi_2(x_2, x_3, t)\,.$$

We will prove below that ϕ is a clean generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration

$$X_1 \times X_3 \times (X_2 \times S \times T) \to X_1 \times X_3\,.$$

The argument is similar to that above: As above C_ϕ is defined by the three sets of equations:

1. $\frac{\partial \phi_1}{\partial s} = 0$
2. $\frac{\partial \phi_2}{\partial t} = 0$
3. $\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_2} = 0.$

Since ϕ_1 and ϕ_2 are transverse generating functions the equations 1 and 2 are an independent set of defining equations for $C_{\phi_1} \times C_{\phi_2}$. As for the equation 3, our assumption that Γ_1 and Γ_2 compose cleanly tells us that the mappings

$$\frac{\partial \phi_1}{\partial x_2} : C_{\phi_1} \to T^* X_2$$

and

$$\frac{\partial \phi_2}{\partial x_2} : C_{\phi_2} \to T^* X_2$$

intersect cleanly. In other words the subset, C_ϕ, of $C_{\phi_1} \times C_{\phi_2}$ defined by the equation, $\frac{\partial \phi}{\partial x_2} = 0$, is a submanifold of $C_{\phi_1} \times C_{\phi_2}$, and its tangent space at each point is defined by the linear equation, $d\frac{\partial \phi}{\partial x_2} = 0$. Thus the set of equations, 1–3, are a *clean* set of defining equations for C_ϕ as a submanifold of $X_1 \times X_3 \times (X_2 \times S \times T)$. In other words ϕ is a clean generating function for $\Gamma_2 \circ \Gamma_1$.

The *excess*, ϵ, of this generating function is equal to the dimension of C_ϕ minus the dimension of $X_1 \times X_3$. One also gets a more intrinsic description of ϵ in terms of the projections of Γ_1 and Γ_2 onto $T^* X_2$. From these projections one gets a map

$$\Gamma_1 \times \Gamma_2 \to T^*(X_2 \times X_2)$$

which, by the cleanness assumption, intersects the conormal bundle of the diagonal cleanly; so its pre-image is a submanifold, $\Gamma_2 \star \Gamma_1$, of $\Gamma_1 \times \Gamma_2$. It's easy to see that

$$\epsilon = \dim \Gamma_2 \star \Gamma_1 - \dim \Gamma_2 \circ \Gamma_1 \, .$$

5.8 Reducing the number of fiber variables.

Let $\Lambda \subset T^* X$ be a Lagrangian manifold and let $\phi \in C^\infty(Z)$ be a generating function for Λ relative to a fibration $\pi : Z \to X$. Let

$$x_0 \in X,$$

let

$$Z_0 := \pi^{-1}(x_0)$$

and let

$$\iota_0 : Z_0 \to Z$$

be the inclusion of the fiber Z_0 into Z. By definition, a point $z_0 \in Z_0$ belongs to C_ϕ if and only if z_0 is a critical point of the restriction $\iota_0^* \phi$ of ϕ to Z_0.

Theorem 31 *If z_0 is a non-degenerate critical point of $\iota_0^*\phi$ then Λ is horizontal at*

$$p_0 = (x_0, \xi_0) = \frac{\partial \phi}{\partial x}(z_0).$$

Moreover, there exists an neighborhood U of x_0 in X and a function $\psi \in C^\infty(U)$ such that

$$\Lambda = \Lambda_\psi$$

on a neighborhood of p_0 and

$$\pi^*\psi = \phi$$

on a neighborhood U' of z_0 in C_ϕ.

Proof. (In local coordinates.) So $Z = X \times \mathbb{R}^k$, $\phi = \phi(x, s)$ and C_ϕ is defined by the k independent equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \dots k. \tag{5.10}$$

Let $z_0 = (x_0, s_0)$ so that s_0 is a non-degenerate critical point of $\iota_0^*\phi$ which is the function

$$s \mapsto \phi(x_0, s)$$

if and only if the Hessian matrix

$$\left(\frac{\partial^2 \phi}{\partial s_i \partial s_j}\right)$$

is of rank k. By the implicit function theorem we can solve equations (5.10) for s in terms of x near (x_0, s_0). This says that we can find a neighborhood U of x_0 in X and a C^∞ map

$$g : U \to \mathbb{R}^k$$

such that

$$g(x) = s \Leftrightarrow \frac{\partial \phi}{\partial s_i} = 0, \ i = 1, \dots, k$$

if (x, s) is in a neighborhood of (x_0, s_0) in Z. So the map

$$\gamma : U \to U \times \mathbb{R}^k, \qquad \gamma(x) = (x, g(x))$$

maps U diffeomorphically onto a neighborhood of (x_0, s_0) in C_ϕ. Consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & C_\phi \\ \downarrow & & \downarrow {\scriptstyle d_X\phi} \\ X & \xleftarrow[\pi_X]{} & \Lambda \end{array}$$

where the left vertical arrow is inclusion and π_X is the restriction to Λ of the projection $T^*X \to X$. From this diagram it is clear that the restriction of π to the image of U in C_ϕ is a diffeomorphism and that Λ is horizontal at p_0. Also

$$\mu := d_X\phi \circ \gamma$$

is a section of Λ over U. Let

$$\psi := \gamma^*\phi.$$

Then

$$\mu = d_X\phi \circ \gamma = d_X\phi \circ \gamma + d_S\phi \circ \gamma = d\phi \circ \gamma$$

since $d_S\phi \circ \gamma \equiv 0$. Also, if $v \in T_xX$ for $x \in U$, then

$$d\psi_x(v) = d\phi_{\gamma(x)}(d\gamma_x(v)) = d\phi_{\gamma(x)}(v, dg_x(v)) = d_X\phi{\circ}\gamma(v)$$

so

$$\langle \mu(x), v\rangle = \langle d\psi_x, v\rangle$$

so

$$\Lambda = \Lambda_\psi$$

over U and from $\pi : Z \to X$ and $\gamma \circ \pi = \text{id}$ on $\gamma(U) \subset C_\phi$ we have

$$\pi^*\psi = \pi^*\gamma^*\phi = (\gamma \circ \pi)^*\phi = \phi$$

on $\gamma(U)$. $\square$

We can apply the proof of this theorem to the following situation: Suppose that the fibration

$$\pi : Z \to X$$

can be factored as a succession of fibrations

$$\pi = \pi_1 \circ \pi_0$$

where

$$\pi_0 : Z \to Z_1 \quad \text{and} \quad \pi_1 : Z_1 \to X$$

are fibrations. Moreover, suppose that the restriction of ϕ to each fiber

$$\pi_0^{-1}(z_1)$$

has a unique non-degenerate critical point $\gamma(z_1)$. The map

$$z_1 \mapsto \gamma(z_1)$$

defines a smooth section

$$\gamma : Z_1 \to Z$$

of π_0. Let

$$\phi_1 := \gamma^* \phi.$$

Theorem 32 *ϕ_1 is a generating function for Λ with respect to π_1.*

Proof. (Again in local coordinates.) We may assume that

$$Z = X \times S \times T$$

and

$$\pi(x,s,t) = x, \quad \pi_0(x,s,t) = (x,s), \quad \pi_1(x,s) = x.$$

The condition for (x,s,t) to belong to C_ϕ is that

$$\frac{\partial \phi}{\partial s} = 0$$

and

$$\frac{\partial \phi}{\partial t} = 0.$$

This last condition has a unique solution giving t as a smooth function of (x,s) by our non-degeneracy condition, and from the definition of ϕ_1 it follows that $(x,s) \in C_{\phi_1}$ if and only if $\gamma(x,s) \in C_\phi$. Furthermore

$$d_X \phi_1(x,s) = d_X \phi(x,s,t)$$

along $\gamma(C_{\phi_1})$. □

For instance, suppose that $Z = X \times \mathbb{R}^k$ and $\phi = \phi(x,s)$ so that $z_0 = (x_0, s_0) \in C_\phi$ if and only if

$$\frac{\partial \phi}{\partial s_i}(x_0, s_0) = 0, \qquad i = 1, \ldots, k.$$

Suppose that the matrix

$$\left(\frac{\partial^2\phi}{\partial s_i \partial s_j}\right)$$

is of rank r, for some $0 < r \leq k$. By a linear change of coordinates we can arrange that the upper left hand corner

$$\left(\frac{\partial^2\phi}{\partial s_i \partial s_j}\right), \quad 1 \leq i, j, \leq r$$

is non-degenerate. We can apply Theorem 32 to the fibration

$$X \times \mathbb{R}^k \to X \times \mathbb{R}^\ell, \qquad \ell = k - r$$

$$(x, s_1, \ldots s_k) \mapsto (x, t_1, \ldots, t_\ell), \qquad t_i = s_{i+r}$$

to obtain a generating function $\phi_1(x,t)$ for Λ relative to the fibration

$$X \times \mathbb{R}^\ell \to X.$$

Thus by reducing the number of variables we can assume that at $z_0 = (x_0, t_0)$

$$\frac{\partial^2\phi}{\partial t_i \partial t_j}(x_0, t_0) = 0, \qquad i, j = 1, \ldots, \ell. \tag{5.11}$$

A generating function satisfying this condition will be said to be **reduced** at (x_0, t_0).

5.9 The existence of generating functions.

In this section we will show that every Lagrangian submanifold of T^*X can be described locally by a generating function ϕ relative to some fibration $Z \to X$.

So let $\Lambda \subset T^*X$ be a Lagrangian submanifold and let $p_0 = (x_0, \xi_0) \in \Lambda$. To simplify the discussion let us temporarily make the assumption that

$$\xi_0 \neq 0. \tag{5.12}$$

If Λ is horizontal at p_0 then we know from Chapter 1 that there is a generating function for Λ near p_0 with the trivial (i.e. no) fibration. If Λ is not horizontal at p_0, we can find a Lagrangian subspace

$$V_1 \subset T_{p_0}(T^*X)$$

which is horizontal and transverse to $T_{p_0}(\Lambda)$.

Indeed, to say that V_1 is horizontal, is to say that it is transverse to the Lagrangian subspace W_1 given by the vertical vectors in the fibration $T^*X \to X$. By the Proposition in §2.2 we know that we can find a Lagrangian subspace which is transversal to both W_1 and $T_{p_0}(\Lambda)$.

Let Λ_1 be a Lagrangian submanifold passing through p_0 and whose tangent space at p_0 is V_1. So Λ_1 is a horizontal Lagrangian submanifold and

$$\Lambda_1 \pitchfork \Lambda = \{p_0\}.$$

In words, Λ_1 intersects Λ transversally at p_0. Since Λ_1 is horizontal, we can find a neighborhood U of x_0 and a function $\phi_1 \in C^\infty(U)$ such that $\Lambda_1 = \Lambda_{\phi_1}$. By our assumption (5.12)

$$(d\phi_1)_{x_0} = \xi_0 \neq 0.$$

So we can find a system of coordinates $x_1 \ldots, x_n$ on U (or on a smaller neighborhood) so that

$$\phi_1 = x_1.$$

Let $\xi_1 \ldots, \xi_n$ be the dual coordinates so that in the coordinate system

$$x_1 \ldots, x_n, \xi_1 \ldots, \xi_n$$

on T^*X the Lagrangian submanifold Λ_1 is described by the equations

$$\xi_1 = 1, \xi_2 = \cdots = \xi_n = 0.$$

Consider the canonical transformation generated by the function

$$\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \tau(x, y) = x \cdot y.$$

The Lagrangian submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ generated by τ is

$$\{(x, y, y, x)\}$$

so the canonical relation is

$$\{(x, \xi, -\xi, x)\}.$$

In other words, it is the graph of the linear symplectic transformation

$$\gamma : (x, \xi) \mapsto (-\xi, x).$$

So $\gamma(\Lambda_1)$ is the cotangent space at $y_0 = (-1, 0, \ldots, 0)$. Since $\gamma(\Lambda)$ is transverse to this cotangent fiber, it follows that $\gamma(\Lambda)$ is horizontal. So in some neighborhood W of y_0 there is a function ψ such that

$$\gamma(\Lambda) = \Lambda_\psi$$

over W. By equation (5.7) we know that

$$\tau^*(x, y) = -\tau(y, x) = -y \cdot x$$

is the generating function for γ^{-1}. Furthermore, near p_0,

$$\Lambda = \gamma^{-1}(\Lambda_\psi).$$

Hence, by Theorem 30 the function

$$\psi_1(x, y) := -y \cdot x + \psi(y) \tag{5.13}$$

is a generating function for Λ relative to the fibration

$$(x, y) \mapsto y.$$

We have proved the existence of a generating function under the auxiliary hypothesis (5.12). However it is easy to deal with the case $\xi_0 = 0$ as well. Namely, suppose that $\xi_0 = 0$. Let $f \in C^\infty(X)$ be such that $df(x_0) \neq 0$. Then

$$\gamma_f : T^*X \to T^*X, \quad (x, \xi) \mapsto (x, \xi + df)$$

is a symplectomorphism and $\gamma_f(p_0)$ satisfies (5.12). We can then form

$$\gamma \circ \gamma_f(\Lambda)$$

which is horizontal. Notice that $\gamma \circ \gamma_f$ is given by

$$(x, \xi) \mapsto (x, \xi + df) \mapsto (-\xi - df, x).$$

If we consider the generating function on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$g(x, z) = x \cdot z + f(x)$$

then the corresponding Lagrangian submanifold is

$$\{(x, z + df, z, x)\}$$

so the canonical relation is

$$\{(x, -z - df, z, x)\}$$

or, setting $\xi = -z - df$ so $z = -\xi - df$ we get

$$\{(x, \xi, -\xi - df, x)\}$$

which is the graph of $\gamma \circ \gamma_f$. We can now repeat the previous argument. So we have proved:

Theorem 33 *Every Lagrangian submanifold of T^*X can be locally represented by a generating function relative to a fibration.*

Let us now discuss generating functions for canonical relations: So let X and Y be manifolds and

$$\Gamma \subset T^*X \times T^*Y$$

a canonical relation. Let $(p_0, q_0) = (x_0, \xi_0, y_0, \eta_0) \in \Gamma$ and assume now that

$$\xi_0 \neq 0, \quad \eta_0 \neq 0. \tag{5.14}$$

We claim that the following theorem holds

Theorem 34 *There exist coordinate systems $(U, x_1, \ldots, x_n)$ about x_0 and $(V, y_1 \ldots, y_k)$ about y_0 such that if*

$$\gamma_U : \ T^*U \to T^*\mathbb{R}^n$$

is the transform

$$\gamma_U(x, \xi) = (-\xi, x)$$

and

$$\gamma_V : T^*V \to T^* * \mathbb{R}^k$$

is the transform

$$\gamma_V(y, \eta) = (-\eta, y)$$

then locally near

$$p_0' := \gamma_U^{-1}(p_0) \quad \textit{and} \quad q_0' := \gamma_V(q_0)$$

the canonical relation

$$\gamma_V^{-1} \circ \Gamma \circ \gamma_U \tag{5.15}$$

is of the form

$$\Gamma_\phi, \quad \phi = \phi(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^k).$$

Proof. Let

$$M_1 := T^*X, \quad M_2 = T^*Y$$

and

$$V_1 := T_{p_0}M_1, \quad V_2 := T_{q_0}M_2, \quad \Sigma := T_{(p_0,q_0)}\Gamma$$

so that Σ is a Lagrangian subspace of

$$V_1^- \times V_2.$$

Let W_1 be a Lagrangian subspace of V_1 so that (in the linear symplectic category)

$$\Sigma(W_1) = \Sigma \circ W_1$$

is a Lagrangian subspace of V_2. Let W_2 be another Lagrangian subspace of V_2 which is transverse to $\Sigma(W_1)$. We may choose W_1 and W_2 to be horizontal subspaces of $T_{p_0}M_1$ and $T_{q_0}M_2$. Then $W_1 \times W_2$ is transverse to Σ in $V_1 \times V_2$ and we may choose a Lagrangian submanifold passing through p_0 and tangent to W_1 and similarly a Lagrangian submanifold passing through q_0 and tangent to W_2. As in the proof of Theorem 33 we can arrange local coordinates $(x_1 \ldots, x_n)$ on X and hence dual coordinates $(x_1, \ldots x_n, \xi_1, \ldots, \xi_n)$ around p_0 such that the Lagrangian manifold tangent to W_1 is given by

$$\xi_1 = -1, \quad \xi_2 = \cdots \xi_n = 0$$

and similarly dual coordinates on $M_2 = T^*Y$ such that the second Lagrangian submanifold (the one tangent to W_2) is given by

$$\eta_1 = -1, \quad \eta_2 = \cdots = \eta_k = 0.$$

It follows that the Lagrangian submanifold corresponding to the canonical relation (5.15) is horizontal and hence is locally of the form Γ_ϕ. $\square$

5.10 The Legendre transformation.

Coming back to our proof of the existence of a generating function for Lagrangian manifolds, let's look a little more carefully at the details of this proof. Let $X = \mathbb{R}^n$ and let $\Lambda \subset T^*X$ be the Lagrangian manifold defined by the fibration, $Z = X \times \mathbb{R}^n \xrightarrow{\pi} X$ and the generating function

$$\phi(x, y) = -x \cdot y + \psi(y) \tag{5.16}$$

where $\psi \in C^\infty(\mathbb{R}^n)$. Then

$$(x, y) \in C_\phi \Leftrightarrow x = \frac{\partial \psi}{\partial y}(y)\,.$$

Recall also that $(x_0, y_0) \in C_\phi \Leftrightarrow$ the function $\phi(x_0, y)$ has a critical point at y_0. Let us suppose this is a *non-degenerate* critical point, i.e., that the matrix

$$\frac{\partial^2 \phi}{\partial y_i \partial y_j}(x_0, y_0) = \frac{\partial \psi}{\partial y_i \partial y_j}(y_0) \tag{5.17}$$

is of rank n. Then there exists a neighborhood $U \ni x_0$ and a function $\psi^* \in C^\infty(U)$ such that

$$\begin{aligned} \psi^*(x) &= \phi(x, y) \text{ at } (x, y) \in C_\phi & (5.18)\\ \Lambda &= \Lambda_\psi & (5.19) \end{aligned}$$

locally near the image $p_0 = (x_0, \xi_0)$ of the map $\frac{\partial \phi}{\partial x} : C_\phi \to \Lambda$. What do these three assertions say? Assertion(5.17) simply says that the map

$$y \to \frac{\partial \psi}{\partial y} \tag{5.20}$$

is a diffeomorphism at y_0. Assertion(5.18) says that

$$\psi^*(x) = -xy + \psi(x) \tag{5.21}$$

at $x = \frac{\partial \psi}{\partial y}$, and assertion(5.19) says that

$$x = \frac{\partial \psi}{\partial y} \Leftrightarrow y = -\frac{\partial \psi^*}{\partial x} \tag{5.22}$$

i.e., the map

$$x \to -\frac{\partial \psi^*}{\partial x} \tag{5.23}$$

is the inverse of the mapping (5.17). The mapping (5.17) is known as the Legendre transform associated with ψ and the formulas (5.21)– (5.23) are the famous inversion formula for the Legendre transform. Notice also that in the course of our proof that (5.21) is a generating function for Λ we proved that ψ is a generating function for $\gamma(\Lambda)$, i.e., locally near $\gamma(p_0)$

$$\gamma(\Lambda) = \Lambda_\psi \,.$$

Thus we've proved that locally near p_0

$$\Lambda_{\psi^*} = \gamma^{-1}(\Lambda_\psi)$$

where

$$\gamma^{-1} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$$

is the transform $(y, \eta) \to (x, \xi)$ where

$$y = \xi \quad \text{and} \quad x = -\eta \,.$$

This identity will come up later when we try to compute the semi-classical Fourier transform of the rapidly oscillating function

$$a(y)e^{i\frac{\psi(y)}{\hbar}} \,,\ a(y) \in C_0^\infty(\mathbb{R}^n) \,.$$

5.11 The Hörmander-Morse lemma.

In this section we will describe some relations between different generating functions for the same Lagrangian submanifold. Our basic goal is to show that if we have two generating functions for the same Lagrangian manifold they can be obtained (locally) from one another by applying a series of "moves", each of a very simple type.

Let Λ be a Lagrangian submanifold of T^*X, and let

$$Z_0 \stackrel{\pi_0}{\to} X, \quad Z_1 \stackrel{\pi_1}{\to} X$$

be two fibrations over X. Let ϕ_1 be a generating function for Λ with respect to $\pi_1 : Z_1 \to X$.

Proposition 11 *If*

$$f : Z_0 \to Z_1$$

is a diffeomorphism satisfying

$$\pi_1 \circ f = \pi_0$$

then

$$\phi_0 = f^*\phi_1$$

is a generating function for Λ with respect to π_0.

Proof. We have $d(\phi_1 \circ f) = d\phi_0$. Since f is fiber preserving,

$$d(\phi_1 \circ f)_{vert} = (d\phi_0)_{vert}$$

so f maps C_{ϕ_0} diffeomorphically onto C_{ϕ_1}. Furthermore, on C_{ϕ_0} we have

$$d\phi_1 \circ f = (d\phi_1 \circ f)_{hor} = (d\phi_0)_{hor}$$

so f conjugates the maps $d_X\phi_i : C_{\phi_i} \to \Lambda, \quad i = 0, 1$. Since $d_X\phi_1$ is a diffeomorphism of C_{ϕ_1} with Λ we conclude that $d_X\phi_0$ is a diffeomorphism of C_{ϕ_0} with Λ, i.e. ϕ_0 is a generating function for Λ. □

Our goal is to prove a result in the opposite direction. So as above let $\pi_i : Z_i \to X, \ i = 0, 1$ be fibrations and suppose that ϕ_0 and ϕ_1 are generating functions for Λ with respect to π_i. Let

$$p_0 \in \Lambda$$

and $z_i \in C_{\phi_i}, \ i = 0, 1$ be the pre-images of p_0 under the diffeomorphism $d\phi_i$ of C_{ϕ_i} with Λ. So

$$d_X\phi_i(z_i) = p_0, \qquad i = 0, 1.$$

Finally let $x_0 \in X$ be given by

$$x_0 = \pi_0(z_0) = \pi_1(z_1)$$

and let $\psi_i, \ i = 0, 1$ be the restriction of ϕ_i to the fiber $\pi_i^{-1}(x_0)$. Since $z_i \in C_{\phi_i}$ we know that z_i is a critical point for ψ_i. Let

$$d^2\psi_i(z_i)$$

be the Hessian of ψ_i at z_i.

Theorem 35 The Hörmander Morse lemma. *If $d^2\psi_0(z_0)$ and $d^2\psi_1(z_1)$ have the same rank and signature, then there exists neighborhood U_0 of z_0 in Z_0 and U_1 of z_1 in Z_1 and a diffeomorphism*

$$f : U_0 \to U_1$$

such that

$$\pi_1 \circ f = \pi_0$$

and

$$\phi_1 \circ f = f^*\phi_1 = \phi_0 + \text{ const.} \quad .$$

Proof. We will prove this theorem in a number of steps. We will first prove the theorem under the additional assumption that Λ is horizontal at p_0. Then we will reduce the general case to this special case.

Assume that Λ is horizontal at $p_0 = (x_0, \xi_0)$. This implies that Λ is horizontal over some neighborhood of x_0. Let S be an open subset of $\mathbb{R}^k$ and

$$\pi : X \times S \to X$$

projection onto the first factor. Suppose that $\phi \in C^\infty(X \times S)$ is a generating function for Λ with respect to π so that

$$d_X\phi : \quad C_\phi \to \Lambda$$

is a diffeomorphism, and let $z_0 \in C_\phi$ be the pre-image of p_0 under this diffeomorphism, i.e.

$$z_0 = (d_X\phi)^{-1}(p_0).$$

We begin by proving that the vertical Hessian of ϕ at z_0 is non-degenerate.

Since Λ is horizontal at p_0 there is a neighborhood U of x_0 $\psi \in C^\infty(U)$ such that

$$d\psi : U \to T^*X$$

maps U diffeomorphically onto a neighborhood of p_0 in Λ. So

$$(d\psi)^{-1} \circ d_X\phi : \ C_\phi \to U$$

is a diffeomorphism. But $(d\psi)^{-1}$ is just the restriction to a neighborhood of p_0 in Λ of the projection $\pi_X : T^*X \to X$. So $\pi_X \circ d_X\phi : C_\phi \to X$ is a diffeomorphism (when restricted to $\pi^{-1}(U)$). But

$$\pi_X \circ d_X\phi = \pi_{|C_\phi}$$

so the restriction of π to C_ϕ is a diffeomorphism. So C_ϕ is horizontal at z_0, in the sense that

$$T_{z_0}C_\phi \cap T_{z_0}S = \{0\}.$$

So we have a smooth map

$$\mathbf{s} : U \to S$$

such that $x \mapsto (x, \mathbf{s}(x))$ is a smooth section of C_ϕ over U. We have

$$d_X\phi = d\phi \quad \text{at all points} \quad (x, \mathbf{s}(x))$$

by the definition of C_ϕ and $d\psi(x) = d_X\phi(x, \mathbf{s}(x)) = d\phi(x, \mathbf{s}(x))$ so

$$\psi(x) = \phi(x, \mathbf{s}(x)) + \text{const.} \ . \tag{5.24}$$

The submanifold $C_\phi \subset Z = X \times S$ is defined by the k equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \ldots, k$$

and hence $T_{z_0}C_\phi$ is defined by the k independent linear equations

$$d\left(\frac{\partial \phi}{\partial s_i}\right) = 0, \qquad i = 1, \ldots, k.$$

A tangent vector to S at z_0, i.e. a tangent vector of the form

$$(0, v), \quad v = (v^1, \ldots v^k)$$

will satisfy these equations if and only if

$$\sum_j \frac{\partial^2 \phi}{\partial s_i \partial s_j} v^j = 0, \qquad i = 1, \ldots, k.$$

But we know that these equations have only the zero solution as no non-zero tangent vector to S lies in the tangent space to C_ϕ at z_0. We conclude that the vertical Hessian matrix

$$d_S^2\phi = \left(\frac{\partial^2 \phi}{\partial s_i \partial s_j}\right)$$

is non-degenerate.

We return to the proof of the theorem under the assumption that that Λ is horizontal at $p_0 = (x_0, \xi_0)$. We know that the vertical Hessians occurring in the statement of the theorem are both non-degenerate, and we are assuming that they are of the same rank. So the fiber dimensions of π_0 and π_1 are the same. So we may assume that $Z_0 = X \times S$ and $Z_1 = X \times S$ where S is an open subset of $\mathbb{R}^k$ and that coordinates have been chosen so that the coordinates of z_0 are $(0, 0)$ as are the coordinates of z_1. We write

$$\mathfrak{s}_0(x) = (x, \mathbf{s}_0(x)), \quad \mathfrak{s}_1(x) = (x, \mathbf{s}_1(x)),$$

where $\mathbf{s}_0$ and $\mathbf{s}_1$ are smooth maps $X \to \mathbb{R}^k$ with

$$\mathbf{s}_0(0) = \mathbf{s}_1(0) = 0.$$

Let us now take into account that the signatures of the vertical Hessians are the same at z_0. By continuity they must be the same at the points $(x, \mathbf{s}_0(x))$ and $(x, \mathbf{s}_1(x))$ for each $x \in U$. So for each fixed $x \in U$ we can make an affine change of coordinates in S and add a constant to ϕ_1 so as to arrange that

1. $\mathbf{s}_0(x) = \mathbf{s}_1(x) = 0$.
2. $\frac{\partial \phi_0}{\partial s_i}(x, 0) = \frac{\partial \phi_1}{\partial s_i}(x, 0),\ i = 1 \ldots, k$.
3. $\phi_0(x, 0) = \phi_1(x, 0)$.
4. $d_S^2 \phi_0(x, 0) = d_S^2 \phi_1(x, 0)$.

We can now apply Morse's lemma with parameters (see §9.14.3 for a proof) to conclude that there exists a fiber preserving diffeomorphism $f : U \times S \to U \times S$ with

$$f^* \phi_1 = \phi_0.$$

This completes the proof of Theorem 35 under the additional hypothesis that Lagrangian manifold Λ is horizontal.

Reduction of the number of fiber variables. Our next step in the proof of Theorem 35 will be an application of Theorem 32. Let $\pi : Z \to X$ be a fibration and ϕ a generating function for Λ with respect to π. Suppose we are in the setup of Theorem 32 which we recall with some minor changes in notation: We suppose that the fibration

$$\pi : Z \to X$$

can be factored as a succession of fibrations

$$\pi = \rho \circ \varrho$$

where

$$\rho : Z \to W \quad \text{and} \quad \varrho : W \to X$$

are fibrations. Moreover, suppose that the restriction of ϕ to each fiber

$$\rho^{-1}(w)$$

has a unique non-degenerate critical point $\gamma(w)$. The map

$$w \mapsto \gamma(w)$$

defines a smooth section

$$\gamma : W \to Z$$

of ρ. Let

$$\chi := \gamma^* \phi.$$

Theorem 32 asserts that χ is a generating function of Λ with respect to ϱ. Consider the Lagrangian submanifold

$$\Lambda_\chi \subset T^* W.$$

This is horizontal as a Lagrangian submanifold of T^*W and ϕ is a generating function for Λ_χ relative to the fibration $\rho : Z \to W$.

Now suppose that we had two fibrations and generating functions as in the hypotheses of Theorem 35 and suppose that they both factored as above with *the same* $\varrho : W \to X$ and the same χ. So we get fibrations $\varrho_0 : Z_0 \to W$ and $\varrho_1 : Z_1 \to W$. We could then apply the above (horizontal) version of Theorem 35 to conclude the truth of the theorem.

Since the ranks of $d^2\psi_0$ and $d^2\psi_1$ at z_0 and z_1 are the same, we can apply the reduction leading to equation (5.11) to each. So by the above argument Theorem 35 will be proved once we prove it for the reduced case.

Some normalizations in the reduced case. We now examine a fibration $Z = X \times S \to S$ and generating function ϕ and assume that ϕ is reduced at $z_0 = (x_0, s_0)$ so all the second partial derivatives of ϕ in the S direction vanish, i. e.

$$\frac{\partial^2 \phi}{\partial s_i \partial s_j}(x_0, s_0) = 0 \quad \forall i, j.$$

This implies that

$$T_{s_0} S \cap T_{(x_0, s_0)} C_\phi = T_{s_0} S.$$

i.e. that

$$T_{s_0} S \subset T_{(x_0, s_0)} C_\phi. \tag{5.25}$$

Consider the map

$$d_X \phi : X \times S \to T^* X, \qquad (x, s) \mapsto d_X \phi(x, s).$$

The restriction of this map to C_ϕ is just our diffeomorphism of C_ϕ with Λ. So the restriction of the differential of this map to any subspace of any tangent space to C_ϕ is injective. By (5.25) the restriction of the differential of this map to $T_{s_0}S$ at (x_0, s_0) is injective. In other words, by passing to a smaller neighborhood of (x_0, s_0) if necessary, we have an embedding

$$\begin{array}{ccc} X \times S & \xrightarrow{d_X\phi} & W \subset T^*X \\ \pi\downarrow & & \downarrow\pi_X \\ X & \xrightarrow[\text{id}]{} & X \end{array}$$

of $X \times S$ onto a subbundle W of T^*X.

Now let us return to the proof of our theorem. Suppose that we have two generating functions ϕ_i, $i = 0, 1$ $X \times S_i \to X$ and both are reduced at the points z_i of C_{ϕ_1} corresponding to $p_0 \in \Lambda$. So we have two embeddings

$$\begin{array}{ccc} X \times S_i & \xrightarrow{d_X\phi_i} & W_i \subset T^*X \\ \pi\downarrow & & \downarrow\pi_X \\ X & \xrightarrow[\text{id}]{} & X \end{array}$$

of $X \times S_i$ onto subbundle W_i of T^*X for $i = 0, 1$. Each of these maps the corresponding C_{ϕ_i} diffeomorphically onto Λ.

Let V be a tubular neighborhood of W_1 in T^*X and $\tau : V \to W_1$ a projection of V onto W_1 so we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\tau} & W_1 \\ \pi_X\downarrow & & \downarrow\pi_X \\ X & \xrightarrow[\text{id}]{} & X \end{array} .$$

Let

$$\gamma := (d_X\phi_1)^{-1} \circ \tau.$$

So we have the diagram

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & X \times S_1 \\ \pi\downarrow & & \downarrow\pi_X \\ X & \xrightarrow[\text{id}]{} & X \end{array}$$

and

$$\gamma \circ d_X \phi_1 = \mathrm{id}\,.$$

We may assume that $W_0 \subset V$ so we get a fiber map

$$g := \gamma \circ d_X \phi_0 \quad g : X \times S_0 \to X \times S_1.$$

When we restrict g to C_{ϕ_0} we get a diffeomorphism of C_{ϕ_0} onto C_{ϕ_1}. By (5.25) we know that

$$T_{s_i} S_i \subset T_{z_i} C_{\phi_i}$$

and so dg_{z_0} maps $T_{s_0} S_0$ bijectively onto $T_{s_1} S_1$. Hence g is locally a diffeomorphism at z_0. So by shrinking X and S_i we may assume that

$$g : X \times S_0 \to X \times S_1$$

is a fiber preserving diffeomorphism. We now apply Proposition 11. So we replace ϕ_1 by $g^*\phi_1$. Then the two fibrations Z_0 and Z_1 are the same and $C_{\phi_0} = C_{\phi_1}$. Call this common submanifold C. Also $d_X\phi_0 = d_X\phi_1$ when restricted to C, and by definition the vertical derivatives vanish. So $d\phi_0 = d\phi_1$ on C, and so by adjusting an additive constant we can arrange that $\phi_0 = \phi_1$ on C.

Completion of the proof. We need to prove the theorem in the following situation:

- $Z_0 = Z_1 = X \times S$ and $\pi_0 = \pi_1$ is projection onto the first factor.
- The two generating functions ϕ_0 and ϕ_1 have the same critical set:
$$C_{\phi_0} = C_{\phi_1} = C.$$
- $\phi_0 = \phi_1$ on C.
- $d_S\phi_i = 0,\ i = 0, 1$ on C and $d_X\phi_0 = d_X\phi_1$ on C.
- $$d\left(\frac{\partial \phi_0}{\partial s_i}\right) = d\left(\frac{\partial \phi_1}{\partial s_i}\right) \text{ at } z_0.$$

We will apply the Moser trick: Let

$$\phi_t := (1-t)\phi_0 + t\phi_1.$$

From the above we know that

- $\phi_t = \phi_0 = \phi_1$ on C.
- $d_S\phi_t = 0$ on C and $d_X\phi_t = d_X\phi_0 = d_X\phi_1$ on C.
- $$d\left(\frac{\partial \phi_t}{\partial s_i}\right) = d\left(\frac{\partial \phi_0}{\partial s_i}\right) = d\left(\frac{\partial \phi_1}{\partial s_i}\right) \text{ at } z_0.$$

So in a sufficiently small neighborhood of Z_0 the submanifold C is defined by the k independent equations

$$\frac{\partial \phi_t}{\partial s_i} = 0, \quad i = 1, \dots k.$$

We look for a vertical (time dependent) vector field

$$v_1 = \sum_i v_i(x,s,t)\frac{\partial}{\partial s_i}$$

on $X \times S$ such that

1. $D_{v_t}\phi_t = -\dot{\phi_t} = \phi_0 - \phi_1$ and
2. $v = 0$ on C.

Suppose we find such a v_t. Then solving the differential equations

$$\frac{d}{dt}f_t(m) = v_t(f_t(m)), \quad f_0(m) = m$$

will give a family of fiber preserving diffeomorphsms (since v_t is vertical) and

$$f_1^*\phi_1 - \phi_0 = \int_0^1 \frac{d}{dt}(f_t^*\phi_t)dt = \int_0^1 f_t^*[D_{v_t}\phi_t + \dot{\phi_t}]dt = 0.$$

So finding a vector field v_t satisfying 1) and 2) will complete the proof of the theorem. Now $\phi_0 - \phi_1$ vanishes to second order on C which is defined by the independent equations $\partial\phi_t/\partial s_i = 0$. So we can find functions

$$w_{ij}(x,s,t)$$

defined and smooth in some neighborhood of C such that

$$\phi_0 - \phi_1 = \sum_{ij} w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial s_i}\frac{\partial \phi_t}{\partial s_j}$$

in this neighborhood. Set

$$v_i(x,s,t) = \sum_i w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial s_j}.$$

Then condition 2) is clearly satisfied and

$$D_{v_t}\phi_t = \sum_{ij} w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial si}\frac{\partial \phi_t}{\partial s_j} = \phi_0 - \phi_1 = -\dot{\phi}$$

as required. □

5.12 Changing the generating function.

We summarize the results of the preceding section as follows: Suppose that $(\pi_1 : Z_1 \to X, \phi_1)$ and $(\pi_2 : Z_2 \to X, \phi_2)$ are two descriptions of the same Lagrangian submanifold Λ of T^*X. Then locally one description can be obtained from the other by applying sequentially "moves" of the following three types:

1. **Adding a constant.** We replace ϕ_1 by $\phi_2 = \phi_1 + c$ where c is a constant.

2. **Equivalence.** There exists a diffeomorphism $g:\ Z_1 \to Z_2$ with

   $$\pi_2 \circ g = \pi_1 \quad \text{and} \quad \text{and} \quad \phi_2 \circ g = \phi_1.$$

3. **Increasing (or decreasing) the number of fiber variables.** Here $Z_2 = Z_1 \times \mathbb{R}^d$ and

   $$\phi_2(z,s) = \phi_1(z) + \frac{1}{2}\langle As, s\rangle$$

   where A is a non-degenerate $d \times d$ matrix.

5.13 The Maslov bundle.

We wish to associate to each Lagrangian submanifold of a cotangent bundle a certain flat line bundle which will be of importance to us when we get to the symbol calculus in Chapter 8. We begin with a review of the Čech-theoretic description of flat line bundles.

5.13.1 The Čech description of locally flat line bundles.

Let Y be a manifold and $\mathbb{U} = \{U_i\}$ be an open cover of Y. Let

$$\mathbb{N}^1 = \{(i,j) | U_i \cap U_j \neq \emptyset\}.$$

A collection of of non-zero complex numbers $\{c_{ij}\}$ is called a (multiplicative) cocycle (relative to the cover $\mathbb{U}$) if

$$c_{ij} \cdot c_{jk} = c_{ik} \quad \text{whenever} \quad U_i \cap U_j \cap U_k \neq \emptyset. \tag{5.26}$$

From this data one constructs a line bundle as follows: One considers the set

$$\amalg_i (U_i \times \mathbb{C})$$

and puts an an equivalence relation on it by declaring that

$$(p_i, a_i) \sim (p_j, a_j) \quad \Leftrightarrow \quad p_i = p_j \in U_i \cap U_j \text{ and } a_i = c_{ij} a_j.$$

Then

$$\mathbb{L} := \amalg_i (U_i \times \mathbb{C}) / \sim$$

is a line bundle over Y. The constant functions

$$U_i \to 1 \in \mathbb{C}$$

form flat local sections of $\mathbb{L}$

$$s_i : U \to \mathbb{L}, \quad p \mapsto [(p, 1)]$$

and thus make $\mathbb{L}$ into a line bundle with flat connection over Y.

Any section s of $\mathbb{L}$ can be written over U_i as $s = f_i s_i$. If v is a vector field on Y, we may define $D_v s$ by

$$D_v s := (D_v f_i) s_i \quad \text{on} \quad U_i.$$

The fact that the transitions between s_i and s_j are constant shows that this is well defined.

5.13.2 The local description of the Maslov cocycle.

We first define the Maslov line bundle $\mathbb{L}_{\text{Maslov}} \to \Lambda$ in terms of a global generating function, and then show that the definition is invariant under change of generating function. We then use the local existence of generating functions to patch the line bundle together globally. Here are the details:

Suppose that ϕ is a generating function for Λ relative to a fibration $\pi : Z \to X$. For each z be a point of the critical set C_ϕ, let $x = \pi(z)$ and let $F = \pi^{-1}(x)$ be the fiber containing z. The restriction of ϕ to the fiber F has a critical point at z. Let $\mathrm{sgn}^{\#}(z)$ be the signature of the Hessian at z of ϕ restricted to F. This gives an integer valued function on C_ϕ:

$$\mathrm{sgn}^{\#} : C_\phi \to \mathbb{Z}, \qquad z \mapsto \mathrm{sgn}^{\#}(z).$$

Notice that since the Hessian can be singular at points of C_ϕ this function can be quite discontinuous.

From the diffeomorphism $\lambda_\phi = d_X\phi$

$$\lambda_\phi : C_\phi \to \Lambda$$

we get a $\mathbb{Z}$ valued function

$$\mathrm{sgn}_\phi := \mathrm{sgn}^\sharp \circ \lambda_\phi^{-1}.$$

Let

$$s_\phi := e^{\frac{\pi i}{4}\,\mathrm{sgn}_\phi}.$$

So

$$s_\phi : \Lambda \to \mathbb{C}^*$$

taking values in the eighth roots of unity.

We define the Maslov bundle $\mathbb{L}_{\mathrm{Maslov}} \to \Lambda$ to be the trivial flat bundle having s_ϕ as its defining flat section. Suppose that (Z_i, π_i, ϕ_i), $\;i = 1, 2$ are two descriptions of Λ by generating functions which differ from one another by one of the three Hörmander moves of Section 5.12. We claim that

$$s_{\phi_1} = c_{1,2}s_{\phi_2} \tag{5.27}$$

for some constant $c_{1,2} \in \mathbb{C}^*$. So we need to check this for the three types of move of Section 5.12. For moves of type 1) and 2), i.e. adding a constant or equivalences this is obvious. For each of these moves there is no change in sgn_ϕ.

For a move of type 3) the $\mathrm{sgn}_1^{\#}$ and $\mathrm{sgn}_2^{\#}$ are related by

$$\mathrm{sgn}_1^{\#} = \mathrm{sgn}_2^{\#} + \text{signature of} A.$$

This proves (5.27), and defines the Maslov bundle when a global generating function exists.

5.13.3 The global definition of the Maslov bundle.

Now consider a general Lagrangian submanifold $\Lambda \subset T^*X$. Cover Λ by open sets U_i such that each U_i *is* defined by a generating function and that generating functions ϕ_i and ϕ_j are obtained from one another by one of the Hörmander moves. We get functions $s_{\phi_i} : U_i \to \mathbb{C}$ such that on every overlap $U_i \cap U_j$

$$s_{\phi_i} = c_{ij} s_{\phi_j}$$

with constants c_{ij} with $|c_{ij}| = 1$. Although the functions s_ϕ might be quite discontinuous, the c_{ij} in (5.27) are constant on $U_i \cap U_j$. On the other hand, the fact that $s_{\phi_i} = c_{ij} s_{\phi_j}$ shows that the cocycle condition (5.26) is satisfied. In other words we get a Cech cocycle on the one skeleton of the nerve of this cover and hence a flat line bundle.

5.13.4 The Maslov bundle of a canonical relation between cotangent bundles.

We have defined the Maslov bundle for any Lagrangian submanifold of any cotangent bundle. If

$$\Gamma \in \operatorname{Morph}(T^*X_1, T^*X_2)$$

is a canonical relation between cotangent bundles, so that Γ is a Lagrangian submanifold of

$$(T^*X_1)^- \times T^*X_2$$

then

$$(\varsigma_1 \times \mathrm{id})(\Gamma)$$

is a Lagrangian submanifold of

$$T^*X_1 \times T^*X_2 = T^*(X_1 \times X_2)$$

and hence has an associated Maslov line bundle. We then use the identification $\varsigma_1 \times \mathrm{id}$ to pull this line bundle back to Γ. In other words, we define

$$\mathbb{L}_{\mathrm{Maslov}}(\Gamma) := (\varsigma_1 \times \mathrm{id})^* \mathbb{L}_{\mathrm{Maslov}}((\varsigma_1 \times \mathrm{id})(\Gamma)). \quad (5.28)$$

5.13.5 Functoriality of the Maslov bundle.

Let X_1, X_2, and X_3 be differentiable manifolds, and let

$$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2) \quad \text{and} \quad \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$$

be cleanly composible canonical relations. Recall that this implies that we have a submanifold

$$\Gamma_2 \star \Gamma_1 \subset T^*X_1 \times T^*X_2 \times T^*X_3$$

and a fibration (4.5)

$$\kappa : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$$

with compact connected fibers. So we can form the line bundle

$$\kappa^*(\mathbb{L}_{\mathrm{Maslov}}(\Gamma_2 \circ \Gamma_1)) \to \Gamma_2 \star \Gamma_1.$$

On the other hand, $\Gamma_2 \star \Gamma_1$ consists of all (m_1, m_2, m_3) with

$$(m_1, m_2) \in \Gamma_1 \quad \text{and} \quad (m_2.m_3) \in \Gamma_2.$$

So we have projections

$$\mathrm{pr}_1 : \ \Gamma_2 \star \Gamma_1 \to \Gamma_1, \qquad (m_1, m_2, m_3) \mapsto (m_1, m_2)$$

and

$$\mathrm{pr}_2 : \ \Gamma_2 \star \Gamma_1 \to \Gamma_2, \qquad (m_1, m_2, m_3) \mapsto (m_2, m_3).$$

So we can also pull the Maslov bundles of Γ_1 and Γ_2 back to $\Gamma_2 \star \Gamma_1$. We claim that

$$\kappa^*\mathbb{L}_{\mathrm{Maslov}}(\Gamma_2 \circ \Gamma_1) \cong \mathrm{pr}_1^*\,\mathbb{L}_{\mathrm{Maslov}}(\Gamma_1) \otimes \mathrm{pr}_2^*\,\mathbb{L}_{\mathrm{Maslov}}(\Gamma_2) \tag{5.29}$$

as line bundles over $\Gamma_2 \star \Gamma_1$.

Proof. We know from Section 5.7 that we can locally choose generating functions ϕ_1 for Γ_1 relative to a fibration

$$X_1 \times X_2 \times S_1 \to X_1 \times S_2$$

and ϕ_2 for Γ_2 relative to a fibration

$$X_2 \times X_3 \times S_2 \to X_2 \times X_3$$

so that

$$\phi = \phi(x_1, x_2, x_3, s_1, s_2) = \phi_1(x_1, x_2, s_1) + \phi_2(x_2, x_3, s_2)$$

is a generating function for $\Gamma_2 \circ \Gamma_1$ relative to the fibration

$$X_1 \times X_3 \times X_2 \times S_1 \times S_2 \to X_1 \times X_3$$

(locally). We can consider the preceding equation as taking place over a neighborhood in $\Gamma_2 \star \Gamma_1$ yielding

$$\operatorname{sgn} \phi = \operatorname{sgn} \phi_1 + \operatorname{sgn} \phi_2. \tag{5.30}$$

We may further restrict our choices of generating functions and neighborhoods for Γ_1 so that the passage from one to the other is given by one of the Hörmander moves, and similarly for Γ_2. But this then gives a cover of $\Gamma_2 \star \Gamma_1$ by open sets giving local flat sections of the line bundles of both sides of (5.29) for which the Čech cocyles clearly yield (5.29). □

We will study the geometry of the Maslov bundle in more detail in Chapter **??**

5.14 Examples of generating functions.

5.14.1 The image of a Lagrangian submanifold under geodesic flow.

Let X be a geodesically convex Riemannian manifold, for example $X = \mathbb{R}^n$. Let f_t denote geodesic flow on X. We know that for $t \neq 0$ a generating function for the symplectomorphism f_t is

$$\psi_t(x, y) = \frac{1}{2t} d(x, y)^2.$$

Let Λ be a Lagrangian submanifold of T^*X. Even if Λ is horizontal, there is no reason to expect that $f_t(\Lambda)$ be horizontal - caustics can develop. But our theorem about the generating function of the composition of two canonical relations will give a generating function for $f_t(\Lambda)$. Indeed, suppose that ϕ is a generating function for Λ relative to a fibration

$$\pi : X \times S \to X.$$

Then

$$\frac{1}{2}d(x,y)^2 + \psi(y,s)$$

is a generating function for $f_t(\Lambda)$ relative to the fibration

$$X \times X \times S \to X, \quad (x,y,s) \mapsto x.$$

5.14.2 The billiard map and its iterates.

Definition of the billiard map.

Let Ω be a bounded open convex domain in $\mathbb{R}^n$ with smooth boundary X. We may identify the tangent space to any point of $\mathbb{R}^n$ with $\mathbb{R}^n$ using the vector space structure, and identify $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ using the standard inner product. Then at any $x \in X$ we have the identifications

$$T_xX \cong T_xX^*$$

using the Euclidean scalar product on T_xX and

$$T_xX = \{v \in \mathbb{R}^n |\ v \cdot n(x) = 0\} \tag{5.31}$$

where $n(x)$ denotes the inward pointing unit normal to X at x. Let $U \subset TX$ denote the open subset consisting of all tangent vectors (under the above identification) satisfying

$$\|v\| < 1.$$

For each $x \in X$ and $v \in T_xX$ satisfying $\|v\| < 1$ let

$$u := v + an(x) \ \text{ where } a := (1 - \|v\|^2)^{\frac{1}{2}}.$$

So u is the unique inward pointing unit vector at x whose orthogonal projection onto T_xX is v.

Consider the ray through x in the direction of u, i.e. the ray

$$x + tu, \quad t > 0.$$

Since Ω is convex and bounded, this ray will intersect X at a unique point y. Let w be the orthogonal projection of u on T_yX. So we have defined a map

$$\mathcal{B} : U \to U, \quad (x,v) \mapsto (y,w)$$

which is known as the **billiard map**.

The generating function of the billiard map.

We shall show that the billiard map is a symplectomorphism by writing down a function ϕ which is its generating function.

Consider the function

$$\psi: \ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ \psi(x,y) = \|x-y\|.$$

This is smooth at all points (x,y), $x \neq y$. Let us compute $d_x\psi(v)$ at such a point (x,y) where $v \in T_xX$.

$$\frac{d}{dt}\psi(x+tv,y)_{|t=0} = \left(\frac{x-y}{\|y-x\|}, v\right)$$

where (,) denotes the scalar product on $\mathbb{R}^n$. Identifying $T\mathbb{R}^n$ with $T^*\mathbb{R}^n$ using this scalar product, we can write that for all $x \neq y$

$$d_x\psi(x,y) = -\frac{y-x}{\|x-y\|}, \qquad d_y\psi(x,y) = \frac{y-x}{\|x-y\|}.$$

If we set

$$u = \frac{y-x}{\|x-y\|}, \quad t = \|x-y\|$$

we have

$$\|u\| = 1$$

and

$$y = x + tu.$$

Let ϕ be the restriction of ψ to $X \times X \subset \mathbb{R}^n \times \mathbb{R}^n$. Let

$$\iota : X \to \mathbb{R}^n$$

denote the embedding of X into $\mathbb{R}^n$. Under the identifications

$$T_x\mathbb{R}^n \cong T_x^*\mathbb{R}^n, \quad T_xX \cong T_x^*X$$

the orthogonal projection

$$T_x^*\mathbb{R}^n \cong T_x\mathbb{R}^n \ni u \mapsto v \in T_xX \cong T_x^*X$$

is just the map

$$d\iota_x^* : T_x^*\mathbb{R}^n \to T_x^*X, \quad u \mapsto v.$$

So

$$v = d\iota_x^* u = d\iota_x^* d_x\psi(x,y) = d_x\phi(x,y).$$

So we have verified the conditions

$$v = -d_x\phi(x,y), \qquad w = d_y\phi(x,y)$$

which say that ϕ is a generating function for the billiard map $\mathcal{B}$.

Iteration of the billiard map.

Our general prescription for the composite of two canonical relations says that a generating function for the composite is given by the sum of generating functions for each (where the intermediate variable is regarded as a fiber variable over the initial and final variables). Therefore a generating function for $\mathcal{B}^n$ is given by the function

$$\phi(x_0, x_1, \ldots, x_n) = \|x_1 - x_0\| + \|x_2 - x_1\| + \cdots + \|x_n - x_{n-1}\|.$$

5.14.3 The classical analogue of the Fourier transform.

We repeat a previous computation: Let $X = \mathbb{R}^n$ and consider the map

$$\mathfrak{F} : T^*X \to T^*X, \quad (x, \xi) \mapsto (-\xi, x).$$

The generating function for this symplectomorphism is

$$x \cdot y.$$

Since the transpose of the graph of a symplectomorphism is the graph of the inverse, the generating function for the inverse is

$$-y \cdot x.$$

So a generating function for the identity is

$$\phi \in C^\infty(X \times X, \times \mathbb{R}^n)$$

$$\phi(x, z, y) = (x - z) \cdot y.$$

Chapter 6

The calculus of $\frac{1}{2}$-densities.

An essential ingredient in our symbol calculus will be the notion of a $\frac{1}{2}$- density on a canonical relation. We begin this chapter with a description of densities of arbitrary order on a vector space, then on a manifold, and then specialize to the study of $\frac{1}{2}$-densities. We study $\frac{1}{2}$-densities on canonical relations in the next chapter.

6.1 The linear algebra of densities.

6.1.1 The definition of a density on a vector space.

Let V be an n-dimensional vector space over the real numbers. A basis $\mathbf{e} = e_1, \ldots, e_n$ of V is the same as an isomorphism $\ell_{\mathbf{e}}$ of $\mathbb{R}^n$ with V according to the rule

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_1 e_1 + \cdots + x_n e_n.$$

We can write this as

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (e_1, \ldots e_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

or even more succinctly as

$$\ell_{\mathbf{e}} : \quad \mathbf{x} \mapsto \mathbf{e} \cdot \mathbf{x}$$

where

$$\mathbf{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{e} = (e_1, \ldots, e_n).$$

The group $Gl(n) = Gl(n, \mathbb{R})$ acts on the set $\mathcal{F}(V)$ of all bases of V according to the rule

$$\ell_{\mathbf{e}} \mapsto \ell_{\mathbf{e}} \circ A^{-1}, \qquad A \in Gl(n)$$

which is the same as the "matrix multiplication"

$$\mathbf{e} \mapsto \mathbf{e} \cdot A^{-1}.$$

This action is effective and transitive:

- If $\mathbf{e} = \mathbf{e} \cdot A^{-1}$ for some basis $\mathbf{e}$ then $A = I$, the identity matrix, and
- Given any two bases $\mathbf{e}$ and $\mathbf{f}$ these exists a (unique) A such that $\mathbf{e} = \mathbf{f} \cdot A$.

We shall use the word **frame** as being synonymous with the word "basis", especially when we want to talk of a basis with a particular property.

Let $\alpha \in \mathbb{C}$ be any complex number. A **density of order** α on V is a function

$$\rho : \ \mathcal{F}(V) \to \mathbb{C}$$

satisfying

$$\rho(\mathbf{e} \cdot A) = \rho(\mathbf{e}) |\det A|^{\alpha} \qquad \forall A \in Gl(n), \ \mathbf{e} \in \mathcal{F}(V). \tag{6.1}$$

We will denote the space of all densities of order α on V by

$$|V|^{\alpha}.$$

This is a one dimensional vector space over the complex numbers. Indeed, if we fix one $\mathbf{f} \in \mathcal{F}(V)$, then every $\mathbf{e} \in \mathcal{F}(V)$ can be written uniquely as $\mathbf{e} = \mathbf{f} \cdot B, \ B \in Gl(n)$. So we may specify $\rho(f)$ to be any complex value and then define $\rho(e)$ to be $\rho(f) \cdot |\det B|^{\alpha}$. It is then easy to check that (6.1) holds. This shows that densities of order α exist, and since we had no choice once we specified $\rho(\mathbf{f})$ we see that the space of densities of order α on V form a one dimensional vector space over the complex numbers.

Let $L : V \to V$ be a linear map. If L is invertible and $\mathbf{e} \in \mathcal{F}(V)$ then $L\mathbf{e} = (Le_1, \dots, Le_n)$ is (again) a basis of V. If we write

$$Le_j = \sum_i L_{ij} e_i$$

then

$$L\mathbf{e} = \mathbf{e}\mathrm{L}$$

where L is the matrix

$$\mathrm{L} := (L_{ij})$$

so if $\rho \in |V|^\alpha$ then

$$\rho(L\mathbf{e}) = |\det L|^\alpha \rho(\mathbf{e}).$$

We can extend this to all L, non necessarily invertible, where the right hand side is 0. So here is an equivalent definition of a density of order α on an n-dimensional real vector space:

A density ρ of order α is a rule which assigns a number $\rho(v_1, \dots, v_n)$ to every n-tuplet of vectors and which satisfies

$$\rho(Lv_1, \dots, Lv_n) = |\det L|^\alpha \rho(v_1, \dots, v_n) \qquad (6.2)$$

for any linear transformation $L : V \to V$. Of course, if the $v_1, \dots, v_n$ are not linearly independent then

$$\rho(v_1, \dots, v_n) = 0.$$

6.1.2 Multiplication.

If $\rho \in |V|^\alpha$ and $\tau \in |V|^\beta$ then we get a density $\rho \cdot \tau$ of order $\alpha + \beta$ given by

$$(\rho \cdot \tau)(\mathbf{e}) = \rho(\mathbf{e})\tau(\mathbf{e}).$$

In other words we have an isomorphism:

$$|V|^\alpha \otimes |V|^\beta \cong |V|^{\alpha+\beta}, \quad \rho \otimes \tau \mapsto \rho \cdot \tau. \qquad (6.3)$$

6.1.3 Complex conjugation.

If $\rho \in |V|^\alpha$ then $\overline{\rho}$ defined by

$$\overline{\rho}(\mathbf{e}) = \overline{\rho(\mathbf{e})}$$

is a density of order $\overline{\alpha}$ on V. In other words we have an anti-linear map

$$|V|^\alpha \to |V|^{\overline{\alpha}}, \qquad \rho \mapsto \overline{\rho}.$$

This map is clearly an anti-linear isomorphism. Combined with (6.3) we get a sesquilinear map

$$|V|^\alpha \otimes |V|^\beta \to |V|^{\alpha+\overline{\beta}}, \rho \otimes \tau \mapsto \rho \cdot \overline{\tau}.$$

We will especially want to use this for the case $\alpha = \beta = \frac{1}{2} + is$ where s is a real number. In this case we get a sesquilinear map

$$|V|^{\frac{1}{2}+is} \otimes |V|^{\frac{1}{2}+is} \to |V|^1. \tag{6.4}$$

6.1.4 Elementary consequences of the definition.

There are two obvious but very useful facts that we will use repeatedly:

1. An element of $|V|^\alpha$ is completely determined by its value on a single basis $\mathbf{e}$.

2. More generally, suppose we are given a subset S of the set of bases on which a subgroup $H \subset Gl(n)$ acts transitively and a function $\rho : S \to \mathbb{C}$ such that (6.1) holds for all $A \in H$. Then ρ extends uniquely to a density of order α on V.

Here are some typical ways that we will use these facts:

Orthonormal frames: Suppose that V is equipped with a scalar product. This picks out a subset $\mathcal{O}(V) \subset \mathcal{F}(V)$ consisting of the orthonormal frames. The corresponding subgroup of $Gl(n)$ is $O(n)$ and every element of $O(n)$ has determinant ± 1. So any density of any order must take on a constant value on orthonormal frames, and item 2 above implies that any constant then determines a density of any order. We have trivialized the space $|V|^\alpha$ for all α. Another way of saying the same thing is that V has a preferred density of order α, namely the density which assigns the value one to any orthonormal frame. The same applies if V has any non-degenerate quadratic form, not necessarily positive definite.

Symplectic frames: Suppose that V is a symplectic vector space, so $n = \dim V = 2d$ is even. This picks out a collection of preferred bases, namely those of the form $e_1, \ldots, e_d,\ f_1, \ldots f_d$ where

$$\omega(e_i, e_j) = 0, \quad \omega(f_i, f_j) = 0. \quad \omega(e_i, f_j) = \delta_{ij}$$

where ω denotes the symplectic form. These are known as the symplectic frames. In this case $H = Sp(n)$ and every element of $Sp(n)$ has determinant one. So again $|V|^\alpha$ is trivialized. Again, another way of saying this is that a symplectic vector space has a preferred density of any order - the density which assigns the value one to any symplectic frame.

Transverse Lagrangian subspaces: Suppose that V is a symplectic vector space and that M and N are Lagrangian subspaces of V with $M \cap N = \{0\}$. *Any* basis $e_1, \ldots e_d$ of M determines a dual basis $f_1, \ldots f_d$ of N according to the requirement that

$$\omega(e_i, f_j) = \delta_{ij}$$

and then $e_1, \ldots e_d, f_1 \ldots f_d$ is a symplectic basis of V. If $C \in Gl(d)$ and we make the replacement

$$\mathbf{e} \mapsto \mathbf{e} \cdot C$$

then we must make the replacement

$$\mathbf{f} \mapsto \mathbf{f} \cdot (C^t)^{-1}.$$

So if ρ is a density of order α on M and τ is a density of order α on N they fit together to get a density of order zero (i.e. a constant) on V according to the rule

$$(\mathbf{e}, \mathbf{f}) = (e_1, \ldots, e_d,\ f_1, \ldots, f_d) \mapsto \rho(\mathbf{e})\tau(\mathbf{f})$$

on frames of the above dual type. The corresponding subgroup of $Gl(n)$ is a subgroup of

$Sp(n)$ isomorphic to $Gl(d)$. So we have a canonical isomorphism

$$|M|^\alpha \otimes |N|^\alpha \cong \mathbb{C}. \qquad (6.5)$$

Using (6.3) we can rewrite this as

$$|M|^\alpha \cong |N|^{-\alpha}.$$

Dual spaces: If we start with a vector space M we can make $M \oplus M^*$ into a symplectic vector space with M and M^* transverse Lagrangian subspaces and the pairing B between M and M^* just the standard pairing of a vector space with its dual space. So making a change in notation we have

$$|V|^\alpha \cong |V^*|^{-\alpha}. \qquad (6.6)$$

Short exact sequences: Let

$$0 \to V' \to V \to V'' \to 0$$

be an exact sequence of linear maps of vector spaces. We can choose a preferred set of bases of V as follows : Let $(e_1, \dots, e_k)$ be a basis of V' and extend it to a basis $(e_1, \dots, e_k, e_{k+1}, \dots e_n)$ of V. Then the images of $e_i,\ i = k+1, \dots n$ under the map $V \to V''$ form a basis of V''. Any two bases of this type differ by the action of an $A \in Gl(n)$ of the form

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$$

so

$$\det A = \det A' \cdot \det A''.$$

This shows that we have an isomorphism

$$|V|^\alpha \cong |V'|^\alpha \otimes |V''|^\alpha \qquad (6.7)$$

for any α.

Long exact sequences Let

$$0 \to V_1 \to V_2 \to \cdots V_k \to 0$$

be an exact sequence of vector spaces. Then using (6.7) inductively we get

$$\bigotimes_{j \text{ even}} |V_j|^\alpha \cong \bigotimes_{j \text{ odd}} |V_j|^\alpha \qquad (6.8)$$

for any α.

6.1.5 Pullback and pushforward under isomorphism.

Let

$$L : V \to W$$

be an isomorphism of n-dimensional vector spaces. If

$$\mathbf{e} = (e_1, \ldots, e_n)$$

is a basis of V then

$$L\mathbf{e} := (Le_1, \ldots, Le_n)$$

is a basis of W and

$$L(\mathbf{e} \cdot A) = (L\mathbf{e}) \cdot A \quad \forall A \in Gl(n).$$

So if $\rho \in |W|^\alpha$ then $L^*\rho$ defined by

$$(L^*\rho)(\mathbf{e}) := \rho(L\mathbf{e})$$

is an element of $|V|^\alpha$. In other words we have a **pullback isomorphism**

$$L^* : |W|^\alpha \to |V|^\alpha, \quad \rho \mapsto L^*\rho.$$

Applied to L^{-1} this gives a **pushforward isomorphism**

$$L_* : |V|^\alpha \to |W|^\alpha, \quad L_* = (L^{-1})^*.$$

6.1.6 Lefschetz symplectic linear transformations.

There is a special case of (6.5) which we will use a lot in our applications, so we will work out the details here. A linear map $L : V \to V$ on a vector space is called **Lefschetz** if it has no eigenvalue equal to 1. Another way of saying this is that $I - L$ is invertible. Yet another way of saying this is the following: Let

$$\operatorname{graph} L \subset V \oplus V$$

be the graph of L so

$$\operatorname{graph} L = \{(v, Lv) \quad v \in V\}.$$

Let

$$\Delta \subset V \oplus V$$

be the diagonal, i.e. the graph of the identity transformation. Then L is Lefschetz if and only if

$$\operatorname{graph} L \cap \Delta = \{0\}. \tag{6.9}$$

Now suppose that V is a symplectic vector space and we consider $V^- \oplus V$ as a symplectic vector space. Suppose also that L is a (linear) symplectic transformation so that $\operatorname{graph} L$ is a Lagrangian subspace of $V^- \oplus V$ as is Δ. Suppose that L is also Lefschetz so that (6.9) holds.

The isomorphism

$$V \to \operatorname{graph} L : \qquad v \mapsto (v, Lv)$$

pushes the canonical α-density on V to an α-density on $\operatorname{graph} L$, namely, if $v_1, \ldots, v_n$ is a symplectic basis of V, then this pushforward α density assigns the value one to the basis

$$((v_1, Lv_1), \ldots, (v_n, Lv_n)) \quad \text{of } \operatorname{graph} L.$$

Let us call this α-density ρ_L. Similarly, we can use the map

$$\operatorname{diag} : V \to \Delta, \qquad v \mapsto (v, v)$$

to push the canonical α density to an α-density ρ_Δ on Δ. So ρ_Δ assigns the value one to the basis

$$((v_1, v_1), \ldots, (v_n, v_n)) \qquad \text{of } \Delta.$$

According to (6.5)

$$|\operatorname{graph} L|^\alpha \otimes |\Delta|^\alpha \cong \mathbb{C}.$$

So we get a number $\langle \rho_L, \rho_\Delta \rangle$ attached to these two α-densities. We claim that

$$\langle \rho_L, \rho_\Delta \rangle = |\det(I - L)|^{-\alpha}. \tag{6.10}$$

Before proving this formula, let us give another derivation of (6.5). Let M and N be subspaces of a symplectic vector space W. (The letter V is currently overworked.) Suppose that $M \cap N = \{0\}$ so that $W = M \oplus N$ as a vector space and so by (6.7) we have

$$|W|^\alpha = |M|^\alpha \otimes |N|^\alpha.$$

We have an identification of $|W|^\alpha$ with $\mathbb{C}$ given by sending

$$|W|^\alpha \ni \rho_W \mapsto \rho_W(\mathbf{w})$$

where $\mathbf{w}$ is any symplectic basis of W. Combining the last two equations gives an identification of $|M|^\alpha \otimes |N|^\alpha$ with $\mathbb{C}$ which coincides with (6.5) in case M and N are Lagrangian subspaces. Put another way, let $\mathbf{w}$ be a symplectic basis of W and suppose that $A \in Gl(\dim W)$ is such that

$$\mathbf{w} \cdot A = (\mathbf{m}, \mathbf{n})$$

where $\mathbf{m}$ is a basis of M and $\mathbf{n}$ is a basis of N. Then the pairing of of $\rho_M \in |M|^\alpha$ with $\rho_N \in |N|^\alpha$ is given by

$$\langle \rho_M, \rho_N \rangle = |\det A|^{-\alpha} \rho_M(\mathbf{m}) \rho_N(\mathbf{n}). \tag{6.11}$$

Now let us go back to the proof of (6.10). If $\mathbf{e}, \mathbf{f} = e_1, \ldots, e_d, f_1 \ldots, f_d$ is a symplectic basis of V then

$$((\mathbf{e}, 0), (0, \mathbf{e}), (-\mathbf{f}, 0), (0, \mathbf{f}))$$

is a symplectic basis of $V^- \oplus V$. We have

$$((\mathbf{e}, 0), (0, \mathbf{e}), (-\mathbf{f}, 0), (0, \mathbf{f})) \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & -I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix} = ((\mathbf{e}, 0), (\mathbf{f}, 0), (0, \mathbf{e}), (0, \mathbf{f}))$$

and

$$\det \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & -I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix} = 1.$$

Let $\mathbf{v}$ denote the symplectic basis $\mathbf{e}, \mathbf{f}$ of V so that we may write

$$((\mathbf{e}, 0), (\mathbf{f}, 0), (0, \mathbf{e}), (0, \mathbf{f})) = ((\mathbf{v}, 0), (0, \mathbf{v}))\,.$$

Write

$$\mathrm{L} v_j = \sum_i L_{ij} v_i, \quad \mathrm{L} = (L_{ij}).$$

Then

$$((\mathbf{v}, 0), (0, \mathbf{v})) \begin{pmatrix} I_d & I_d \\ \mathrm{L} & I_d \end{pmatrix} = ((\mathbf{v}, \mathrm{L}\mathbf{v}), (\mathbf{v}, \mathbf{v}))\,.$$

So taking

$$A = \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & -I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} I_n & I_n \\ \mathrm{L} & I_n \end{pmatrix}$$

we have

$$((\mathbf{e}, 0), (0, \mathbf{e}), (-\mathbf{f}, 0), (0, \mathbf{f}))\, A = ((\mathbf{v}, L\mathbf{v}), (\mathbf{v}, \mathbf{v}))\,.$$

So using this A in (6.11) proves (6.10) since

$$\det A = \det \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & -I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix} \det \begin{pmatrix} I_n & I_n \\ \mathrm{L} & I_n \end{pmatrix} = \det(I_n - L).$$

6.2 Densities on manifolds.

Let $E \to X$ be a real vector bundle. We can then consider the complex line bundle

$$|E|^\alpha \to X$$

whose fiber over $x \in X$ is $|E_x|^\alpha$. The formulas of the preceding section apply pointwise.

We will be primarily interested in the tangent bundle TX. So $|TX|^\alpha$ is a complex line bundle which we will call the α-density bundle and a smooth section of $|TX|^\alpha$ will be called a smooth α-**density** or a **density of order** α.

Examples.

- Let $X = \mathbb{R}^n$ with its standard coordinates and hence the standard vector fields

  $$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}.$$

  This means that at each point $p \in \mathbb{R}^n$ we have a preferred basis

  $$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

  We let

  $$dx^\alpha$$

denote the α-density which assigns, at each point p, the value 1 to the above basis. So the most general smooth α-density on $\mathbb{R}^n$ can be written as

$$u \cdot dx^\alpha$$

or simply as

$$udx^\alpha$$

where u is a smooth function.

- Let X be an n-dimensional Riemannian manifold. At each point p we have a preferred family of bases of the tangent space - the orthonormal bases. We thus get a preferred density of order α - the density which assigns the value one to each orthonormal basis at each point.

- Let X be an n-dimensional orientable manifold and Ω a nowhere vanishing n-form on X. Then we get an α-density according to the rule: At each $p \in X$ assign to each basis $e_1, \ldots, e_n$ of T_pX the value

$$|\Omega(e_1, \ldots, e_n)|^\alpha.$$

We will denote this density by

$$|\Omega|^\alpha.$$

- As a special case of the preceding example, if M is a symplectic manifold of dimension $2d$ with symplectic form ω, take

$$\Omega = \omega \wedge \cdots \wedge \omega \qquad d \ \text{ factors.}$$

So every symplectic manifold has a preferred α-density for any α.

6.2.1 Multiplication of densities.

If μ is an α density and ν is a β density the we can multiply them (pointwise) to obtain an $(\alpha + \beta)$-density $\mu \cdot \nu$. Similarly, we can take the complex conjugate of an α-density to obtain an $\overline{\alpha}$-density.

6.2.2 Support of a density.

Since a density is a section of a line bundle, it makes sense to say that a density *is* or *is not* zero at a point. The **support** of a density is defined to be the closure of the set of points where it is *not* zero.

6.3 Pull-back of a density under a diffeomorphism.

If

$$f : X \to Y$$

is a diffeomorphism, then we get, at each $x \in X$, a linear isomorphism

$$df_x : T_xX \to T_{f(x)}Y.$$

A density ν of order α on Y assigns a density of order α (in the sense of vector spaces) to each T_yY which we can then pull back using df_x to obtain a density of order α on X. We denote this pulled back density by $f^*\nu$. For example, suppose that

$$\nu = |\Omega|^\alpha$$

for an n-form Ω on Y (where $n = \dim Y$). Then

$$f^*|\Omega|^\alpha = |f^*\Omega|^\alpha \tag{6.12}$$

where the $f^*\Omega$ occurring on right hand side of this equation is the usual pull-back of forms.

As an example, suppose that X and Y are open subsets of $\mathbb{R}^n$, then

$$dx^\alpha = |dx_1 \wedge \cdots \wedge dx_n|^\alpha, \qquad |dy|^\alpha = |dy_1 \wedge \cdots \wedge dy_n|^\alpha$$

and

$$f^*(dy_1 \wedge \cdots \wedge dy_n) = \det J(f) dx_1 \wedge \cdots \wedge dx_n$$

where $J(f)$ is the Jacobian matrix of f. So

$$f^*dy^\alpha = |\det J(f)|^\alpha dx^\alpha. \tag{6.13}$$

Here is a second application of (6.12). Let $f_t : X \to X$ be a one-parameter group of diffeomorphisms generated by a vector field v, and let ν be a density of order α on X. As usual, we define the Lie derivative $D_v\nu$ by

$$D_v\nu := \frac{d}{dt} f_t^*\nu_{|t=0}.$$

If $\nu = |\Omega|^\alpha$ then

$$D_v\nu = \alpha D_v|\Omega| \cdot |\Omega|^{\alpha-1}$$

and if X is oriented, then we can identify $|\Omega|$ with Ω on oriented bases, so

$$D_v|\Omega| = D_v\Omega = di(v)\Omega$$

on oriented bases. For example,

$$D_v dx^{\frac{1}{2}} = \frac{1}{2}(\mathrm{div}\ v)dx^{\frac{1}{2}} \tag{6.14}$$

where

$$\mathrm{div}\ v = \frac{\partial v_1}{\partial x_1}+\cdots+\frac{\partial v_n}{\partial x_n} \quad \text{if} \quad v = v_1\frac{\partial}{\partial x_1}+\cdots+v_n\frac{\partial}{\partial x_n}.$$

6.4 Densities of order 1.

If we set $\alpha = 1$ in (6.13) we get

$$f^*dy = |\det J(f)|dx$$

or, more generally,

$$f^*(udy) = (u\circ f)|\det J(f)|dx$$

which is the change of variables formula for a multiple integral. So if ν is a density of order one of compact support which is supported on a coordinate patch $(U, x_1, \ldots, x_n)$, and we write

$$\nu = gdx$$

then

$$\int \nu := \int_U gdx$$

is independent of the choice of coordinates. If ν is a density of order one of compact support we can use a partition of unity to break it into a finite sum of densities of order one and of compact support contained in coordinate patches

$$\nu = \nu_1 + \cdots + \nu_r$$

and $\int_X \nu$ defined as

$$\int_X \nu := \int \nu_1 + \cdots + \int \nu_r$$

is independent of all choices. In other words densities of order one (usually just called densities) are objects which can be integrated (if of compact support). Furthermore, if

$$f : X \to Y$$

is a diffeomorphism, and ν is a density of order one of compact support on Y, we have the general "change of variables formula"

$$\int_X f^*\nu = \int_Y \nu. \tag{6.15}$$

Suppose that α and β are complex numbers with

$$\alpha + \overline{\beta} = 1.$$

Suppose that μ is a density of order α and ν is a density of order β on X and that one of them has compact support. Then $\mu \cdot \overline{\nu}$ is a density of order one of compact support. So we can form

$$\langle \mu, \nu \rangle := \int_X \mu\overline{\nu}.$$

So we get an intrinsic sesquilinear pairing between the densities of order α of compact support and the densities of order $1 - \overline{\alpha}$.

6.5 The principal series representations of Diff(X).

So if $s \in \mathbb{R}$, we get a pre-Hilbert space structure on the space of smooth densities of compact support of order $\frac{1}{2} + is$ given by

$$(\mu, \nu) := \int_X \mu\overline{\nu}.$$

If $f \in \mathrm{Diff}(X)$, i.e. if $f : X \to X$ is a diffeomorphism, then

$$(f^*\mu, f^*\nu) = (\mu, \nu)$$

and

$$(f \circ g)^* = g^* \circ f^*.$$

Let $\mathfrak{H}_s$ denote the completion of the pre-Hilbert space of densities of order $\frac{1}{2} + is$. The Hilbert space $\mathfrak{H}_s$ is known as the **intrinsic Hilbert space of order** s. The map

$$f \mapsto (f^{-1})^*$$

is a representation of Diff (X) on the space of densities or order $\frac{1}{2}+is$ which extends by completion to a unitary representation of Diff (X) on $\mathfrak{H}_s$. This collection of representations (parametrized by s) is known as the principal series of representations.

If we take $S = S^1 = \mathbb{PR}^1$ and restrict the above representations of Diff(X) to $G = PL(2, \mathbb{R})$ we get the principal series of representations of G.

We will concentrate on the case $s = 0$, i.e. we will deal primarily with densities of order $\frac{1}{2}$.

6.6 The push-forward of a density of order one by a fibration.

There is an important generalization of the notion of the integral of a density of compact support: Let

$$\pi : Z \to X$$

be a *proper* fibration. Let μ be a density of order one on Z. We are going to define

$$\pi_* \mu$$

which will be a density of order one on X. We proceed as follows: for $x \in X$, let

$$F = F_x := \pi^{-1}(x)$$

be the fiber over x. Let $z \in F$. We have the exact sequence

$$0 \to T_z F \to T_z Z \stackrel{d\pi_z}{\to} T_x X \to 0$$

which gives rise to the isomorphism

$$|T_z F| \otimes |T_x X| \cong |T_z Z|.$$

The density μ thus assigns to each z in the manifold F an element of

$$|T_z F| \otimes |T_x X|.$$

In other words, on the manifold F it is a density of order one with values in the fixed one dimensional vector space $|T_xX|$. Since F is compact, we can integrate this density over F to obtain an element of $|T_xX|$. As we do this for all x, we have obtained a density of order one on X.

Let us see what the operation $\mu \mapsto \pi_*\mu$ looks like in local coordinates. Let us choose local coordinates $(U, x_1, \ldots, x_n, s_1 \ldots, s_d)$ on Z and coordinates $y_1, \ldots, y_n$ on X so that

$$\pi : (x_1, \ldots, x_n, s_1, \ldots, s_d) \mapsto (x_1, \ldots, x_n).$$

Suppose that μ is supported on U and we write

$$\mu = udxds = u(x_1, \ldots, x_n, s_1 \ldots, s_d)dx_1 \ldots dx_n ds_1 \ldots ds_d.$$

Then

$$\pi_*\mu = \left(\int u(x_1, \ldots, x_n, s_1, \ldots, s_d) ds_1 \ldots ds_d \right) dx_1 \ldots dx_n. \tag{6.16}$$

In the special case that X is a point, $\pi_*\mu = \int_Z \mu$. Also, Fubini's theorem says that if

$$W \xrightarrow{\rho} Z \xrightarrow{\pi} X$$

are fibrations with compact fibers then

$$(\pi \circ \rho)_* = \pi_* \circ \rho_*. \tag{6.17}$$

In particular, if μ is a density of compact support on Z with $\pi : Z \to X$ a fibration then $\pi_*\mu$ is defined and

$$\int_X \pi_*\mu = \int_Z \mu. \tag{6.18}$$

If f is a C^∞ function on X of compact support and $\pi : Z \to X$ is a proper fibration then $\pi^* f$ is constant along fibers and (6.18) says that

$$\int_Z \pi^* f \mu = \int_X f \pi_* \mu. \tag{6.19}$$

In other words, the operations

$$\pi^* : C_0^\infty(X) \to C_0^\infty(Z)$$

and

$$\pi_* : C^\infty(|TZ|) \to C^\infty(|TX|)$$

are transposes of one another.

Chapter 7

The Enhanced Symplectic "Category".

Suppose that M_1, M_2, and M_3 are symplectic manifolds, and that

$$\Gamma_2 \in \mathrm{Morph}(M_2, M_3) \quad \text{and} \qquad \Gamma_1 \in \mathrm{Morph}(M_1, M_2)$$

are canonical relations which can be composed in the sense of Chapter 4. Let ρ_1 be a $\frac{1}{2}$-density on Γ_1 and ρ_2 a $\frac{1}{2}$-density on Γ_2. The purpose of this chapter is to define a $\frac{1}{2}$-density $\rho_2 \circ \rho_1$ on $\Gamma_2 \circ \Gamma_1$ and to study the properties of this composition. In particular we will show that the composition

$$(\Gamma_2, \rho_2) \times (\Gamma_1, \rho_1) \mapsto (\Gamma_2 \circ \Gamma_1, \rho_2 \circ \rho_1)$$

is associative when defined, and that the axioms for a "category" are satisfied.

7.1 The underlying linear algebra.

We recall some definitions from Section 3.4: Let V_1, V_2 and V_3 be symplectic vector spaces and let $\Gamma_1 \subset V_1^- \times V_2$ and $\Gamma_2 \subset V_2^- \times V_3$ be linear canonical relations. We let

$$\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$$

consist of all pairs $((x,y),(y',z))$ such that $y=y'$, and let
$$\tau:\ \Gamma_1\times\Gamma_2\to V_2$$
be defined by
$$\tau(\gamma_1,\gamma_2):=\pi(\gamma_1)-\rho(\gamma_2)$$
so that $\Gamma_2\star\Gamma_1$ is determined by the exact sequence (3.6)
$$0\to\Gamma_2\star\Gamma_1\to\Gamma_1\times\Gamma_2\stackrel{\tau}{\to}V_2\to\operatorname{Coker}\tau\to 0.$$
We also defined
$$\alpha:\Gamma_2\star\Gamma_1\to\Gamma_2\circ\Gamma_1$$
by (3.9):
$$\alpha:(x,y,y,z)\mapsto(x,z).$$
Then $\ker\alpha$ consists of those $(0,v,v,0)\in\Gamma_2\star\Gamma_1$ and we can identify $\ker\alpha$ as a subspace of V_2. We proved that relative to the symplectic structure on V_2 we have (3.13):
$$\ker\alpha=(\operatorname{Im}\tau)^\perp$$
as subspaces of V_2. We are going to use (3.13) to prove

Theorem 36 *There is a canonical isomorphism*
$$|\Gamma_1|^{\frac12}\otimes|\Gamma_2|^{\frac12}\cong|\ker\alpha|\otimes|\Gamma_2\circ\Gamma_1|^{\frac12}. \tag{7.1}$$

Proof. It follows from (3.13) that we have an identification
$$(V_2/\ker\alpha)\sim(V_2/(\operatorname{Im}\tau)^\perp)\sim(\operatorname{Im}\tau)^*.$$
From the short exact sequence
$$0\to\ker\alpha\to V_2\to V_2/\ker\alpha\to 0$$
we get an isomorphism
$$|V_2|^{\frac12}\sim|\ker\alpha|^{\frac12}\otimes|V_2/\ker\alpha|^{\frac12}$$
and from the fact that V_2 is a symplectic vector space we have a canonical trivialization $|V_2|^{\frac12}\cong\mathbb{C}$. Therefore
$$|\ker\alpha|^{\frac12}\cong|V_2/\ker\alpha|^{-\frac12}.$$

But since $(V_2/\ker\alpha) \cong (\operatorname{Im}\tau)^*$ we obtain an identification

$$|\ker\alpha|^{\frac{1}{2}} \cong |\operatorname{Im}\tau|^{\frac{1}{2}}. \tag{7.2}$$

From the exact sequence (3.6) we obtain the short exact sequence

$$0 \to \Gamma_2 \star \Gamma_1 \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} \operatorname{Im}\tau \to 0$$

which gives an isomorphism

$$|\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}} \cong |\Gamma_2 \star \Gamma_1|^{\frac{1}{2}} \otimes |\operatorname{Im}\tau|^{\frac{1}{2}}.$$

From the short exact sequence

$$0 \to \ker\alpha \to \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1 \to 0$$

we get the isomorphism

$$|\Gamma_2 \star \Gamma_1|^{\frac{1}{2}} \cong |\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \otimes |\ker\alpha|^{\frac{1}{2}}.$$

Putting these two isomorphisms together and using (7.2) gives (7.1). □

7.1.1 Transverse composition of $\frac{1}{2}$ densities.

Let us consider the important special case of (7.1) where σ is surjective and so $\ker\alpha = 0$. Then we have a short exact sequence

$$0 \to \Gamma_2 \star \Gamma_1 \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \to 0$$

and an isomorphism

$$\alpha : \Gamma_2 \star \Gamma_1 \cong \Gamma_2 \circ \Gamma_1$$

and so (7.1) becomes

$$|\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \cong |\Gamma_1 \times \Gamma_2|^{\frac{1}{2}}. \tag{7.3}$$

So if we are given $\frac{1}{2}$-densities σ_1 on Γ_1 and σ_2 on Γ_2 we obtain a $\frac{1}{2}$-density $\sigma_2 \circ \sigma_1$ on $\Gamma_2 \circ \Gamma_1$.

Let us work out this "composition" explicitly in the case that Γ_2 is the graph of an isomorphism

$$S : V_2 \to V_3.$$

Then $\rho : \Gamma_2 \to V_2$ is an isomorphism, and so we can identify $\frac{1}{2}$-densities on Γ_2 with $\frac{1}{2}$-densities on V_2.

Let us choose σ_2 to be the $\frac{1}{2}$-density on Γ_2 which is identified with the canonical $\frac{1}{2}$-density on V_2. So if $2d_2 = \dim V_2 = \dim V_3$ and $u_1, \dots, u_{2d_2}$ is a symplectic basis of V_2, then σ_2 assigns the value one to the basis

$$(u_1, Su_1), \dots, (u_{2d_2}, Su_{2d_2})$$

of Γ_2.

Let $2d_1 = \dim V_1$ and let

$$(e_1, f_1), \dots (e_{d_1+d_2}, f_{d_1+d_2})$$

be a basis of Γ_1. Then

$$(e_1, Sf_1), \dots (e_{d_1+d_2}, Sf_{d_1+d_2})$$

is a basis of $\Gamma_2 \circ \Gamma_1$. Under our identification of $\Gamma_2 \circ \Gamma_1$ with $\Gamma_2 \star \Gamma_1$ (which is a subspace of $\Gamma_1 \times \Gamma_2$) this is identified with the basis

$$[(e_1, f_1), (f_1, Sf_1)], \dots, [(e_{d_1+d_2}, f_{d_1+d_2}), (f_{d_1+d_2}), Sf_{d_1+d_2})]$$

of $\Gamma_2 \star \Gamma_1$. The space $\{0\} \times \Gamma_2$ is complementary to $\Gamma_2 \star \Gamma_1$ in $\Gamma_1 \times \Gamma_2$ and the basis

$$[(e_1, f_1), (f_1, Sf_1)], \dots, [(e_{d_1+d_2}, f_{d_1+d_2}), (f_{d_1+d_2}), Sf_{d_1+d_2})],$$

$$[(0,0), (u_1, Su_1)], \dots, [(0,0), (u_{2d_2}, Su_{2d_2})]$$

differs from the basis

$$[(e_1, f_f), (0,0)], \dots, [((e_{d_1+d_2}, f_{d_1+d_2}), (0,0)],$$

$$[(0,0), (u_1, Su_1)], \dots, [(0,0), (u_{2d_2}, Su_{2d_2})]$$

by multiplication by a matrix of the form

$$\begin{pmatrix} I & * \\ 0 & I \end{pmatrix}.$$

We conclude that

Proposition 12 *If Γ_2 is the graph of a symplectomorphism $S : V_2 \to V_3$ and $\sigma_2 \in |\Gamma_2|^{\frac{1}{2}}$ is identified with the canonical $\frac{1}{2}$-density on V_2, then $\sigma_2 \circ \sigma_1$ is given by $(\mathrm{id} \times S)_* \sigma_1$ under the isomorphism $\mathrm{id} \times S$ of Γ_1 with $\Gamma_2 \circ \Gamma_1$. In particular, if $S = \mathrm{id}$ then $\sigma_2 \circ \sigma_1 = \sigma_1$.*

7.2 Half densities and clean canonical compositions.

Let M_1, M_2, M_3 be symplectic manifolds and let $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ be canonical relations. Let

$$\pi : \Gamma_1 \to M_2,\ \pi(m_1, m_2) = m_2, \quad \rho : \Gamma_2 \to M_2,\ \rho(m_2, m_3) = m_2,$$

and $\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$ the fiber product:

$$\Gamma_2 \star \Gamma_1 = \{(m_1, m_2, m_3) | (m_1, m_2) \in \Gamma_1,\ \ (m_2, m_3) \in \Gamma_2\}.$$

Let

$$\alpha : \Gamma_2 \star \Gamma_1 \to M_1 \times M_3, \quad \alpha(m_1, m_2, m_3) = (m_1, m_3).$$

The image of α is the composition $\Gamma_2 \circ \Gamma_1$.

Recall that we say that Γ_1 and Γ_2 intersect cleanly if the maps ρ and π intersect cleanly. If π and ρ intersect cleanly then their fiber product $\Gamma_2 \star \Gamma_1$ is a submanifold of $\Gamma_1 \times \Gamma_2$ and the arrows in the exact square

$$\begin{array}{ccc} \Gamma_2 \star \Gamma_1 & \longrightarrow & \Gamma_1 \\ \Big\downarrow & & \Big\downarrow \pi \\ \Gamma_2 & \underset{\rho}{\longrightarrow} & M_2 \end{array}$$

are smooth maps. Furthermore the differentials of these maps at any point give an exact square of the corresponding linear canonical relations. In particular, α is of constant rank and $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relation. If we further assume that

1. α is proper and

2. the level sets of α are connected and simply connected,

then $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold of $M_1^- \times M_2$ and

$$\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$$

is a fiber map with proper fibers. So our key identity (7.1) holds at the tangent space level: Let $m = (m_1, m_2, m_3) \in \Gamma_2 \star \Gamma_1$ and $q = \alpha(m) \in \Gamma_2 \circ \Gamma_1$ and let $F_q = \alpha^{-1}(q)$ be the fiber of α passing through m. We get an isomorphism

$$|T_m F_q| \otimes |T_q(\Gamma_2 \circ \Gamma_1)|^{\frac{1}{2}} \cong |T_{m_1,m_2}\Gamma_1|^{\frac{1}{2}} \otimes |T_{(m_2,m_3)}\Gamma_2|^{\frac{1}{2}}. \tag{7.4}$$

This means that if we are given half densities ρ_1 on Γ_1 and ρ_2 on Γ_2 we get a half density on $\Gamma_2 \circ \Gamma_1$ by integrating the expression obtained from the left hand side of the above isomorphism over the fiber. This gives us the composition law for half densities. Once we establish the associative law and the existence of the identity we will have have *enhanced* our symplectic category so that now the morphisms consist of pairs (Γ, ρ) where Γ is a canonical relation and where ρ is a half density on Γ.

Notice that if the composition $\Gamma_2 \circ \Gamma_1$ is transverse, then integration is just pointwise evaluation as in Section 7.1.1. In particular, we may apply Proposition 12 pointwise if Γ_2 is the graph of a symplectomorphism. In particular, if $\Gamma_2 = \Delta(X_2)$ is the diagonal in $X_2 \times X_2$ and we use the canonical $\frac{1}{2}$-density σ_Δ coming from the identification of $\Delta(X_2)$ with the symplectic manifold X_2 with its canonical $\frac{1}{2}$-density, then $(\Delta(X_2), \sigma_\Delta) \circ (\Gamma_1 \sigma_1) = (\Gamma_1, \sigma_1)$. This shows that $(\Delta(X_2), \sigma_\Delta)$ acts as the identity for composition on the left at X_2, and using the involutive structure (see below) implies that it is also an identity for composition on the right. This establishes the existence of the identity. For the associative law, we use the trick of of reducing the associative law for composition to the associative law for direct product as in Section 3.3.2:

7.3 Rewriting the composition law.

We will rewrite the composition law in the spirit of Sections 3.3.2 and 4.4: If $\Gamma \subset M^- \times M$ is the graph of a symplectomorphism, then the projection of Γ onto the first factor is a diffeomorphism. The symplectic form on M determines a canonical $\frac{1}{2}$-density on M, and hence on Γ. In particular, we can apply this fact to the identity map, so $\Delta \subset M^- \times M$ carries a canonical $\frac{1}{2}$-density. Hence, the submanifold

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x, y, y, z, x, z)\} \subset M_1 \times M_2 \times M_2 \times M_3 \times M_1 \times M_3$$

as in (4.6) carries a canonical $\frac{1}{2}$-density $\tau_{1,2,3}$. Then we know that

$$\Gamma_2 \circ \Gamma_1 = \tilde{\Delta}_{M_1,M_2,M_3} \circ (\Gamma_1 \times \Gamma_2)$$

and it is easy to check that

$$\rho_2 \circ \rho_1 = \tau_{123} \circ (\rho_1 \times \rho_2).$$

Similarly,

$$(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = \tilde{\Delta}_{M_1,M_2,M_3,M_4} \circ (\Gamma_1 \times \Gamma_2 \times \Gamma_3)$$

and $\tilde{\Delta}_{M_1,M_2,M_3,M_4}$ carries a canonical $\frac{1}{2}$-density $\tau_{1,2,3,4}$ with

$$(\rho_3 \circ \rho_2) \circ \rho_1 = \rho_3 \circ (\rho_2 \circ \rho_1) = \tau_{1.2.3.4} \circ (\rho_1 \times \rho_2 \times \rho_3).$$

This establishes the associative law.

7.4 Enhancing the category of smooth manifolds and maps.

Let X and Y be smooth manifolds and $E \to X$ and $F \to Y$ be vector bundles. According to Atiyah and Bott, a morphism from $E \to X$ to $F \to Y$ consists of a smooth map

$$f : X \to Y$$

and a section

$$r \in C^\infty(\mathrm{Hom}(f^*F, E)).$$

We described the finite set analogue of this concept in Section 3.3.5. If s is a smooth section of $F \to Y$ then we get a smooth section of $E \to X$ via

$$(f, r)^* s(x) := r(s(f(x)), \qquad x \in X.$$

We want to specialize this construction of Atiyah-Bott to the case where E and F are the line bundles of $\frac{1}{2}$-densities on the tangent bundles. So we say that r is an enhancement of the smooth map $f : X \to Y$ or that (f, r) is an enhanced smooth map if r is a smooth section of the line bundle

$$\mathrm{Hom}(|f^*TY|^{\frac{1}{2}}, |TX|^{\frac{1}{2}}).$$

The composition of two enhanced maps

$$(f, r) : (E \to X) \to (F \to Y) \quad \text{and} \quad (g, r') : (F \to Y) \to (G \to Z)$$

is $(g \circ f, r \circ r')$ where, for $\tau \in |T_{g(f(x))}Z|^{\frac{1}{2}}$

$$(r \circ r')(\tau) = r(r'(\tau)).$$

We thus obtain a category whose objects are the line bundles of $\frac{1}{2}$-densities on the tangent bundles of smooth manifolds and whose morphisms are enhanced maps.

If ρ is a $\frac{1}{2}$-density on Y and (f, r) is an enhanced map then we get a $\frac{1}{2}$-density on X by the Atiyah-Bott rule

$$(f, r)^*\rho(x) = r(\rho(f(x)) \in |T_x X|^{\frac{1}{2}}.$$

Then we know that the assignment $(f, r) \mapsto (f, r)^*$ is functorial. We now give some examples of enhancement of particular kinds of maps:

7.4.1 Enhancing an immersion.

Suppose $f : X \to Y$ is an immersion. We then get the conormal bundle $N_f^* X$ whose fiber at x consists of all covectors $\xi \in T^*_{f(x)}Y$ such that $df_x^* \xi = 0$. We have the exact sequence

$$0 \to T_x X \stackrel{df_x}{\to} T_{f(x)}Y \to N_x Y \to 0.$$

Here $N_x Y$ is *defined* as the quotient $T_{f(x)}Y/df_x(T_x X)$. The fact that f is an immersion is the statement that df_x is injective. The space $(N_f^* X_x)$ is the dual space of $N_x Y$. From the exact sequence above we get the isomorphism

$$|T_{f(x)}Y|^{\frac{1}{2}} \cong |N_x Y|^{\frac{1}{2}} \otimes |T_x X|^{\frac{1}{2}}.$$

So

$$\mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_x X|^{\frac{1}{2}}) \cong |T_x X|^{\frac{1}{2}} \otimes |T_{f(x)}Y|^{-\frac{1}{2}} \cong |N_x Y|^{-\frac{1}{2}} \cong |(N_f^* X)_x|^{\frac{1}{2}}.$$

Conclusion. Enhancing an immersion is the same as giving a section of $|N_f^* X|^{\frac{1}{2}}$.

7.4.2 Enhancing a fibration.

Suppose that $\pi : Z \to X$ is a submersion. If $z \in Z$, let V_z denote the tangent space to the fiber $\pi^{-1}(x)$ at z where $x = \pi(z)$. Thus V_z is the kernel of $d\pi_z : T_zZ \to T_{\pi(z)}X$. So we have an exact sequence

$$0 \to V_z \to T_zZ \to T_{\pi(z)}X \to 0$$

and hence the isomorphism

$$|T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}} \otimes |T_{\pi(z)}X|^{\frac{1}{2}}.$$

So

$$\mathrm{Hom}(|T_{\pi(z)}X|^{\frac{1}{2}}, |T_zZ|^{\frac{1}{2}}) \cong |T_{\pi(z)}X|^{-\frac{1}{2}} \otimes |T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}}. \tag{7.5}$$

Conclusion. Enhancing a fibration is the same as giving a section of $|V|^{\frac{1}{2}}$ where V denotes the vertical sub-bundle of the tangent bundle, i.e. the sub-bundle tangent to the fibers of the fibration.

7.4.3 The pushforward via an enhanced fibration.

Suppose that $\pi : Z \to X$ is a fibration with compact fibers and r is an enhancement of π so that r is given by a section of the line-bundle $|V|^{\frac{1}{2}}$ as we have just seen. Let ρ be a $\frac{1}{2}$-density on Z. From the isomorphism

$$|T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}} \otimes |T_{\pi(z)}X|^{\frac{1}{2}}$$

we can regard ρ as section of $|V|^{\frac{1}{2}} \otimes \pi^*|TX|^{\frac{1}{2}}$ and hence

$$r \cdot \rho$$

is a section of $|V| \otimes \pi^*|TX|^{\frac{1}{2}}$. Put another way, for each $x \in X$, $r \cdot \rho$ gives a density (of order one) on $\pi^{-1}(x)$ with values in the fixed vector space $|T_xX|^{\frac{1}{2}}$. So we can integrate this density of order one over the fiber to obtain

$$\pi_*(r \cdot \rho)$$

which is a $\frac{1}{2}$-density on X. If the enhancement r of π is understood, we will denote the push-forward of the $\frac{1}{2}$-density ρ simply by

$$\pi_*\rho.$$

We have the obvious variants on this construction if π is not proper. We can construct $\pi_*(r \cdot \rho)$ if either r or ρ are compactly supported in the fiber direction.

An enhanced fibration $\pi = (\pi, r)$ gives a pull-back operation π^* from half densities on X to $\frac{1}{2}$-densities on Z. So if μ is a $\frac{1}{2}$-density on X and ν is a $\frac{1}{2}$-density on Z then

$$\nu \cdot \pi^* \mu$$

is a density on Z. If μ is of compact support and if ν is compactly supported in the fiber direction, then $\nu \cdot \pi^* \mu$ is a density (of order one) of compact support on Z which we can integrate over Z. We can also form

$$(\pi_* \nu) \cdot \mu.$$

which is a density (of order one) which is of compact support on X. It follows from Fubini's theorem that

$$\int_Z \nu \cdot \pi^* \mu = \int_X (\pi_* \nu) \cdot \mu.$$

7.5 Enhancing a map enhances the corresponding canonical relation.

Let $f : X \to Y$ be a smooth map. We can enhance this map by giving a section r of $\mathrm{Hom}(|TY|^{\frac{1}{2}}, |TX|^{\frac{1}{2}})$. On the other hand, we can construct the canonical relation

$$\Gamma_f \in \mathrm{Morph}(T^*X, T^*Y)$$

as described in Section 4.7. Enhancing this canonical relation amounts to giving a $\frac{1}{2}$-density ρ on Γ_f. In this section we show how the enhancement r of the map f gives rise to a $\frac{1}{2}$-density on Γ_f.

Recall (4.9) which says that

$$\Gamma_f = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1), \quad \xi_1 = df^*_{x_1} \xi_2\}.$$

From this description we see that Γ_f is a vector bundle over X whose fiber over $x \in X$ is $T^*_{f(x)}Y$. So at each point $z = (x, \xi_1, y, \eta) \in \Gamma_f$ we have the isomorphism

$$|T_z \Gamma_f|^{\frac{1}{2}} \cong |T_x X|^{\frac{1}{2}} \otimes |T_\eta (T^*_{f(x)} Y)|^{\frac{1}{2}}.$$

But $(T^*_{f(x)}Y)$ is a vector space, and at any point η in a vector space W we have a canonical identification of $T_\eta W$ with W. So at each $z \in \Gamma_f$ we have an isomorphism

$$|T_z\Gamma_f|^{\frac{1}{2}} \cong |T_xX|^{\frac{1}{2}} \otimes |T_\eta(T^*_{f(x)}Y)|^{\frac{1}{2}} = \mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_xX|^{\frac{1}{2}})$$

and at each x, $r(x)$ is an element of $\mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_xX|^{\frac{1}{2}})$. So r gives rise to a $\frac{1}{2}$-density on Γ_f,

I still need to write up the functoriality of this relation.

7.6 The involutive structure of the enhanced symplectic "category".

Recall that if $\Gamma \in \mathrm{Morph}(M_1, M_2)$ then we defined $\Gamma^\dagger \in (M_2, M_1)$ be

$$\Gamma^\dagger = \{(y, x) | (x, y) \in \Gamma\}.$$

We have the switching diffeomorphism

$$s : \Gamma^\dagger \to \Gamma, \qquad (y, x) \mapsto (x, y),$$

and so if ρ is a $\frac{1}{2}$-density on Γ then $s^*\rho$ is a $\frac{1}{2}$-density on $\Gamma^\dagger$. We define

$$\rho^\dagger = \overline{s^*\rho}. \tag{7.6}$$

Starting with an enhanced morphism (Γ, ρ) we define

$$(\Gamma, \rho)^\dagger = (\Gamma^\dagger, \rho^\dagger).$$

We show that $\dagger : (\Gamma, \rho) \mapsto (\Gamma, \rho)^\dagger$ satisfies the conditions for a involutive structure. Since $s^2 = \mathrm{id}$ it is clear that $\dagger^2 = \mathrm{id}$. If $\Gamma_2 \in \mathrm{Morph}(M_2, M_1)$ and $\Gamma_1 \in \mathrm{Morph}(M_1, M_2)$ are composible morphsims, we know that the composition of (Γ_2, ρ_2) with (Γ_1, ρ_1) is given by

$$(\tilde{\Delta}_{M_1,M_2,M_3}, \tau_{123}) \circ (\Gamma_1 \times \Gamma_2, \rho_1 \times \rho_2).$$

where

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x, y, y, z, x, z) | x \in M_1, y \in M_2, z \in M_3\}$$

and τ_{123} is the canonical (real) $\frac{1}{2}$-density arising from the symplectic structures on M_1, M_2 and M_3. So

$$s : (\Gamma_2 \circ \Gamma_1)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger \to \Gamma_2 \circ \Gamma_1$$

is given by applying the operator S switching x and z

$$S : \tilde{\Delta}_{M_3,M_2,M_1} \to \tilde{\Delta}_{M_1,M_2,M_3},$$

applying the switching operators $s_1 : \Gamma_1^\dagger \to \Gamma_1$ and $s_2 : \Gamma_2^\dagger \to \Gamma_2$ and also switching the order of Γ_1 and Γ_2. Pull-back under switching the order of Γ_1 and Γ_2 sends $\rho_1 \times \rho_2$ to $\rho_2 \times \rho_1$, applying the individual s_1^* and s_2^* and taking complex conjugates sends $\rho_2 \times \rho_1$ to $\rho_2^\dagger \times \rho_1^\dagger$. Also

$$S^* \tau_{123} = \tau_{321}$$

and τ_{321} is real. Putting all these facts together shows that

$$((\Gamma_2, \rho_2) \circ (\Gamma_1, \rho_1))^\dagger = (\Gamma_1, \rho_1)^\dagger \circ (\Gamma_2, \rho_2)^\dagger$$

proving that $\dagger$ satisfies the conditions for a involutive structure.

Let M be an object in our "category", i.e. a symplectic manifold. A "point" of M in our enchanced "category" will consist of a Lagrangian submanifold $\Lambda \subset M$ thought of as an element of Morph(pt., M) (in $\mathcal{S}$) together with a $\frac{1}{2}$-density on Λ. If (Λ, ρ) is such a point, then $(\Lambda, \rho)^\dagger = (\Lambda^\dagger, \rho^\dagger)$ where we now think of the Lagrangian submanifold $\Lambda^\dagger$ as an element of Morph(M, pt.).

Suppose that (Λ_1, ρ_1) and (Λ_2, ρ_2) are "points" of M and that $\Lambda_2^\dagger$ and Λ_1 are composible. Then $\Lambda_2^\dagger \circ \Lambda_1$ in $\mathcal{S}$ is an element of Morph(pt., pt.) which consists of a (single) point. So in our enhanced "category" $\tilde{\mathcal{S}}$

$$(\Lambda_2, \rho_2)^\dagger (\Lambda_1, \rho_1)$$

is a $\frac{1}{2}$-density on a point, i.e. a complex number. We will denote this number by

$$\langle (\Lambda_1, \rho_1), (\Lambda_2, \rho_2) \rangle .$$

7.6.1 Computing the pairing $\langle (\Lambda_1, \rho_1), (\Lambda_2, \rho_2) \rangle$.

This is, of course, a special case of the computation of Section 7.2, where $\Gamma_2 \circ \Gamma_1$ is a point.

The first condition that $\Lambda_2^\dagger$ and Λ_1 be composible is that Λ_1 and Λ_2 intersect cleanly as submanifolds of M. Then the F of (7.4) is $F = \Lambda_1 \cap \Lambda_2$ so (7.4) becomes

$$|T_pF| = |T_p(\Lambda_1 \cap \Lambda_2)| \cong |T_p\Lambda_1|^{\frac{1}{2}} \otimes |T_p\Lambda_2|^{\frac{1}{2}} \quad (7.7)$$

and so ρ_1 and $\overline{\rho_2}$ multiply together to give a density $\rho_1\overline{\rho_2}$ on $\Lambda_1 \cap \Lambda_2$. A second condition on composibility requires that $\Lambda_1 \cap \Lambda_2$ be compact. The pairing is thus

$$\langle(\Lambda_1, \rho_1), (\Lambda_2, \rho_2)\rangle = \int_{\Lambda_1 \cap \Lambda_2} \rho_1\overline{\rho_2}. \quad (7.8)$$

7.6.2 † and the adjoint under the pairing.

In the category of whose objects are Hilbert spaces and whose morphisms are bounded operators, the adjoint $A^\dagger$ of a operator $A : H_1 \to H_2$ is defined by

$$\langle Av, w\rangle_2 = \langle v, A^\dagger w\rangle_1, \quad (7.9)$$

for all $v \in H_1, w \in H_2$ where $\langle\ ,\ \rangle_i$ denotes the scalar product on $H_i,\ i = 1, 2$. This can be given a more categorical interpretation as follows: A vector u in a Hilbert space H determines and is determined by a bounded linear map from $\mathbb{C}$ to H,

$$z \mapsto zu.$$

In other words, if we regard $\mathbb{C}$ as the pt. in the category of Hilbert spaces, then we can regard $u \in H$ as an element of of Morph(pt., H). So if $v \in H$ we can regard $v^\dagger$ as an element of Morph(H, pt.) where

$$v^\dagger(u) = \langle u, v\rangle.$$

So if we regard † as the primary operation, then the scalar product on each Hilbert space is determined by the preceding equation - the right hand side is *defined* as being equal to the left hand side. Then equation (7.9) is a consequence of the associative law and the laws $(A \circ B)^\dagger = B^\dagger \circ A^\dagger$ and $\dagger^2 = \mathrm{id}.$. Indeed

$$\langle Av, w\rangle_2 := w^\dagger \circ A \circ v = (A^\dagger \circ w)^\dagger \circ v =: \langle v, A^\dagger w\rangle_1.$$

So once we agree that a $\frac{1}{2}$-density on pt. is just a complex number, we can conclude that the analogue of (7.9) holds in our enhanced category $\tilde{\mathcal{S}}$: If

(Λ_1, ρ_1) is a "point " of M_1 in our enhanced category, and if (Λ_2, ρ_2) is a "point " of M_2 and if $(\Gamma, \tau) \in \mathrm{Morph}(M_1, M_2)$ then (assuming that the various morphisms are composible) we have

$$\langle ((\Gamma, \tau) \circ (\Lambda_1, \rho_1), (\Lambda_2, \rho_2) \rangle_2 = \left\langle (\Lambda_1, \rho_1), ((\Gamma, \tau)^\dagger \circ (\Lambda_2, \rho_2) \right\rangle_1 . \tag{7.10}$$

7.7 The symbolic distributional trace.

We consider a family of symplectomorphisms as in Section 4.10.7 and follow the notation there. In particular we have the family $\Phi : M \times S \to S$ of symplectomorphisms and the associated moment Lagrangian

$$\Gamma := \Gamma_\Phi \subset M \times M^- \times T^*S.$$

7.7.1 The $\frac{1}{2}$-density on Γ.

Since M is symplectic it has a canonical $\frac{1}{2}$ density. So if we equip S with a half density ρ_S we get a $\frac{1}{2}$ density on $M \times M^- \times S$ and hence a $\frac{1}{2}$ density ρ_Γ making Γ into a morphism

$$(\Gamma, \rho_\Gamma) \in \mathrm{Morph}(M^- \times M, T^*S)$$

in our enhanced symplectic category.

Let $\Delta \subset M^- \times M$ be the diagonal. The map

$$M \to M^- \times M \qquad m \mapsto (m, m)$$

carries the canonical $\frac{1}{2}$-density on M to a $\frac{1}{2}$-density, call it ρ_Δ on Δ enhancing Δ into a morphism

$$(\Delta, \rho_\Delta) \in \mathrm{Morph}(\mathrm{pt.}., M^- \times M).$$

The generalized trace in our enhanced symplectic "category".

Suppose that Γ and Δ are composable. Then we get a Lagrangian submanifold

$$\Lambda = \Gamma \circ \Delta$$

and a $\frac{1}{2}$-density

$$\rho_\Lambda := \rho_\Gamma \circ \rho_\Delta$$

on Λ. The operation of passing from $F : M \times S \to M$ to (Λ, ρ_Λ) can be regarded as the symbolic version of the distributional trace operation in operator theory.

7.7.2 Example: The symbolic trace.

Suppose that we have a single symplectomorphism $f : M \to M$ so that S is a point as is T^*S. Let

$$\Gamma = \Gamma_f = \text{graph } f = \{(m, f(m)),\ m \in M\}$$

considered as a morphism from $M \times M^-$ to a point. Suppose that Γ and Δ intersect transversally so that $\Gamma \cap \Delta$ is discrete. Suppose, in fact, that it is finite. We have the $\frac{1}{2}$-densities ρ_Δ on $T_m\Delta$ and $T_m\Gamma$ at each point m of of $\Gamma \cap \Delta$. Hence, by (6.10), the result is

$$\sum_{m \in \Delta \cap \Gamma} |\det(I - df_m)|^{-\frac{1}{2}}\,. \tag{7.11}$$

7.7.3 General transverse trace.

Let S be arbitrary. We examine the meaning of the hypothesis that that the inclusion $\iota : \Delta \to M \times M$ and the projection $\Gamma \to M \times M$ be transverse.

Since Γ is the image of $(G, \Phi) : M \times S \to M \times M \times T^*S$, the projection of Γ onto $M \times M$ is just the image of the map G given in (4.38). So the transverse composibility condition is

$$G \pitchfork \Delta. \tag{7.12}$$

The fiber product of Γ and Δ can thus be identified with the "fixed point submanifold" of $M \times S$:

$$\mathfrak{F} := \{(m, s) | f_s(m) = m\}.$$

The transversality assumption guarantees that this is a submanifold of $M \times S$ whose dimension is equal to $\dim S$. The transversal version of our composition law for morphisms in the category $\mathcal{S}$ asserts that

$$\Phi : \mathfrak{F} \to T * S$$

is a Lagrangian immersion whose image is

$$\Lambda = \Gamma \circ \Delta.$$

Let us assume that $\mathfrak{F}$ is connected and that Φ is a Lagrangian imbedding. (More generally we might want to assume that $\mathfrak{F}$ has a finite number of connected components and that Φ restricted to each of these components is an imbedding. Then the discussion below would apply separately to each component of $\mathfrak{F}$.)

Let us derive some consequences of the transversality hypothesis $G \pitchfork \Delta$. By the Thom transverslity theorem, there exists an open subset

$$S_O \subset S$$

such that for every $s \in S_O$, the map

$$g_s : M \to M \times M, \qquad g_s(m) = G(m,s) = (mf_s(m))$$

is transverse to Δ. So for $s \in S_O$,

$$g_s^{-1}(\Delta) = \{m_i(s), i = 1, \ldots, r\}$$

is a finite subset of M and the m_i depend smoothly on $s \in S_O$. For each i, $\Phi(m_i(s)) \in T_s^*S$ then depends smoothly on $s \in S_O$. So we get one forms

$$\mu_i := \Phi(m_i(s)) \tag{7.13}$$

parametrizing open subsets Λ_i of Λ. Since Λ is Lagrangian, these one forms are closed. So if we assume taht $H^1(S_O) = \{0\}$, we cad write

$$\mu_i = d\psi_i$$

for $\psi_i \in C^\infty(S_O)$ and

$$\Lambda_i = \Lambda_{\psi_i}.$$

The maps

$$S_O \to \Lambda_i, \quad s \mapsto (s, d\psi_i(s))$$

map S_O diffeomorphically onto Λ_i. The pull-backs of the $\frac{1}{2}$-density $\rho_\Lambda = \rho_\Gamma \circ \rho_\Delta$ under these maps can be written as

$$h_i \rho_S$$

where ρ_S is the $\frac{1}{2}$-density we started with on S and where the h_i are the smooth functions

Victor: details here?

$$h_i(s) = \left|\det(I - df_{m_i})\right|^{-\frac{1}{2}}. \tag{7.14}$$

In other words, on the generic set S_O where g_s is transverse to Δ, we can compute the symbolic trace $h(s)$ of g_s as in the preceding section. At points not in S_O, the "fixed points coalesce" so that g_s is no longer transverse to Δ and the individual g_s no longer have a trace as individual maps. Nevertheless, the parametrized family of maps have a trace as a $\frac{1}{2}$-density on Λ which need not be horizontal over points of S which are not in S_O.

7.7.4 Example: Periodic Hamiltonian trajectories.

Let (M, ω) be a symplectic manifold and

$$H : M \to \mathbb{R}$$

a proper smooth function with no critical points. Let $v = v_H$ be the corresponding Hamiltonian vector field, so that

$$i(v)\omega = -dH.$$

The fact that H is proper implies that v generates a global one parameter group of transformations, so we get a Hamiltonian action of $\mathbb{R}$ on M with Hamiltonian H, so we know that the function Φ of (4.32) (determined up to a constant) can be taken to be

$$\Phi : M \times \mathbb{R} \to T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}, \qquad \Phi(m, t) = (t, H(m)).$$

The fact that $dH_m \neq 0$ for any m implies that the vector field v has no zeros.

Notice that in this case the transversality hypothesis of the previous example is never satisfied. For if it were, we could find a dense set of t for which $\exp tv : M \to M$ has isolated fixed points. But if m is fixed under $\exp tv$ then every point on the orbit $(\exp sv)m$ of m is also fixed under $\exp tv$ and we know that this orbit is a curve since v has no zeros.

So the best we can do is assume clean intersection: Our Γ in this case is

$$\Gamma = \{m, (\exp sv)m, s, H(m))\}.$$

If we set $f_s = \exp sv$ we write this as

$$\Gamma = \{(m, f_s(m), s, H(m))\}.$$

The assumption that the maps $\Gamma \to M \times M$ and

$$\iota : \Delta \to M \times M$$

intersect cleanly means that the fiber product

$$X = \{(m, s) \in M \times \mathbb{R} | f_s(m) = m\}$$

and that its tangent space at (m, s) is

$$\{(v, c) \in T_m M \times \mathbb{R} | v = (df_s)_m(v) + cv(m) \quad (7.15)$$

since

$$dF_{(m,s)}\left(v, c\frac{\partial}{\partial t}\right) = (df_s)_m(v) + cv(m).$$

Chapter 8

Oscillatory $\frac{1}{2}$-densities.

Let (Λ, ψ) be an exact Lagrangian submanifold of T^*X. Let

$$k \in \mathbb{Z}.$$

The plan of this chapter is to associate to (Λ, ψ) and to k a space

$$I^k(X, \Lambda, \psi)$$

of rapidly oscillating $\frac{1}{2}$-densities on X and to study the properties of these spaces. If Λ is horizontal with

$$\Lambda = \Lambda_\phi, \quad \psi \in C^\infty(X),$$

and

$$\psi = \phi \circ (\pi_X)_{|\Lambda}$$

this space will consist of $\frac{1}{2}$-densities of the form

$$\hbar^k a(x, \hbar) e^{i\frac{\phi(x)}{\hbar}} \rho_0$$

where ρ_0 is a fixed non-vanishing $\frac{1}{2}$-density on X and where

$$a \in C^\infty(X \times \mathbb{R}).$$

In other words, so long as $\Lambda = \Lambda_\phi$ is horizontal and $\psi = \phi \circ (\pi_X)_{|\Lambda}$, our space will consist of the $\frac{1}{2}$-densities we studied in Chapter 1.

As we saw in Chapter 1, one must take into account, when solving hyperbolic partial differential equations, the fact that caustics develop as a result of the

Hamiltonian flow applied to initial conditions. So we will need a more general definition. We will make a more general definition in terms of a general generating function relative to a fibration, and then show that the class of oscillating $\frac{1}{2}$-densities on X that we obtain this way is independent of the choice of generating functions.

This will imply that we can associate to every exact canonical relation between cotangent bundles (and every integer k) a class of (oscillatory) integral operators which we will call the semi-classical Fourier operators associated to the canonical relation. We will find that if we have two transversally composible canonical relations, the composition of their semi-classical Fourier operators is a semi-classical Fourier operator associated to the composition of the relations. We will then develop a symbol calculus for these operators and their composition.

In order not to overburden the notation, we will frequently write Λ instead of (Λ, ψ). But a definite choice of ψ will always be assumed. So, for example, we will write $I^k(X, \Lambda)$ instead of $I^k(X, (\Lambda, \psi))$ for the class of $\frac{1}{2}$-densities that we will introduce over the next few sections.

8.1 Definition of $I^k(X, \Lambda)$ in terms of a generating function.

Let $\pi : Z \to X$ be a fibration which is enhanced in the sense of Section 7.4.2. Recall that this means that we are given a smooth section r of $|V|^{\frac{1}{2}}$ where V is the vertical sub-bundle of the tangent bundle of Z. We will assume that r vanishes nowhere. If ν is a $\frac{1}{2}$-density on Z which is of compact support in the vertical direction, then recall from Section 7.4.3 that we get from this data a push-forward $\frac{1}{2}$-density $\pi_*\nu$ on X.

Now suppose that ϕ is a global generating function for (Λ, ψ) with respect to π. Recall that this means that we have fixed the arbitrary constant in ϕ so that

$$\psi(x, \xi) = \phi(z)$$

if $d\phi_z = \pi_z^*\xi$ where $\pi(z) = x$. See the discussion following equation (4.58). Let

$$d := \dim Z - \dim X.$$

We define $I_0^k(X,\Lambda,\phi)$ to be the space of all compactly supported $\frac{1}{2}$-densities on X of the form

$$\mu = \hbar^{k-\frac{d}{2}}\pi_*\left(ae^{i\frac{\phi}{\hbar}}\tau\right) \tag{8.1}$$

where $a = a(z,\hbar)$

$$a \in C_0^\infty(Z \times \mathbb{R})$$

and where τ is a nowhere vanishing $\frac{1}{2}$-density on Z. Then define $I^k(X,\Lambda,\phi)$ to consist of those $\frac{1}{2}$-densities μ such that $\rho\mu \in I_0^k(X,\Lambda,\phi)$ for every $\rho \in C_0^\infty(X)$.

It is clear that $I^k(X,\Lambda,\phi)$ does not depend on the choice of the enhancement r of π or on the choice of τ.

8.1.1 Local description of $I^k(X,\Lambda,\phi)$.

Suppose that $Z = X \times S$ where S is an open subset of $\mathbb{R}^d$ and π is projection onto the first factor. We may choose our fiber $\frac{1}{2}$-density to be the Euclidean $\frac{1}{2}$-density $ds^{\frac{1}{2}}$ and τ to be $\tau_0 \otimes ds^{\frac{1}{2}}$ where τ_0 is a nowhere vanishing $\frac{1}{2}$-density on X. Then $\phi = \phi(x,s)$ and (8.1) becomes the oscillating integral

$$\left(\int_S a(x,s,\hbar)e^{i\frac{\phi}{\hbar}}ds\right)\tau_0. \tag{8.2}$$

8.1.2 Independence of the generating function.

Let $\pi_i : Z_i \to X$, ϕ_i be two fibrations and generating functions for the same Lagrangian submanifold $\Lambda \subset T^*X$. We wish to show that $I^k(X,\Lambda,\phi_1) = I^k(X,\Lambda,\phi_2)$. By a partition of unity, it is enough to prove this locally. According to Section 5.12, since the constant is fixed by (4.58), it is enough to check this for two types of change of generating functions, 1) equivalence and 2) increasing the number of fiber variables. Let us examine each of the two cases:

Equivalence.

There exists a diffeomorphism $g: Z_1 \to Z_2$ with

$$\pi_2 \circ g = \pi_1 \qquad \text{and} \qquad \phi_2 \circ g = \phi_1.$$

Let us fix a non-vanishing section r of the vertical $\frac{1}{2}$-density bundle $|V_1|^{\frac{1}{2}}$ of Z_1 and a $\frac{1}{2}$-density τ_1 on Z_1. Since g is a fiber map, these determine vertical $\frac{1}{2}$-densities and $\frac{1}{2}$-densities g_*r and $g_*\tau_1$ on Z_2. If $a \in C_0^\infty(Z_2 \times \mathbb{R})$ then the change of variables formula for an integral implies that

$$\pi_{2,*} a e^{i\frac{\phi_2}{\hbar}} g_*\tau_1 = \pi_{1,*} g^* a e^{i\frac{\phi_1}{\hbar}} \tau_1$$

where the push-forward $\pi_{2,*}$ on the left is relative to g_*r and the push-forward on the right is relative to r. $\square$

Increasing the number of fiber variables.

We may assume that $Z_2 = Z_1 \times S$ where S is an open subset of $\mathbb{R}^m$ and

$$\phi_2(z, s) = \phi_1(z) + \frac{1}{2}\langle As, s\rangle$$

where A is a symmetric non-degenerate $m \times m$ matrix. We write Z for Z_1. If d is the fiber dimension of Z then $d+m$ is the fiber dimension of Z_2. Let r be a vertical $\frac{1}{2}$-density on Z so that $r \otimes ds^{\frac{1}{2}}$ is a vertical $\frac{1}{2}$-density on Z_2. Let τ be a $\frac{1}{2}$ density on Z so that $\tau \otimes ds^{\frac{1}{2}}$ is a $\frac{1}{2}$-density on Z_2. We want to consider the expression

$$\hbar^{k-\frac{d+m}{2}} \pi_{2*} a_2(z, s, \hbar) e^{i\frac{\phi_2(z,s)}{\hbar}} (\tau \otimes ds^{\frac{1}{2}}).$$

Let $\pi_{2,1} : Z \times S \to Z$ be projection onto the first factor so that

$$\pi_{2*} = \pi_{1*} \circ \pi_{2,1*}$$

and the operation $\pi_{2,1*}$ sends

$$a_2(z, s, \hbar) e^{i\frac{\phi_2}{\hbar}} \tau \otimes ds^{\frac{1}{2}} \mapsto b(z, \hbar) e^{i\frac{\phi_1}{\hbar}} \tau$$

where

$$b(z, \hbar) = \int a_2(z, s, \hbar) e^{\frac{\langle As, s\rangle}{2\hbar}} ds.$$

We now apply the Lemma of Stationary Phase (see Chapter 10) to conclude that

$$b(z,\hbar) = \hbar^m a_1(z,\hbar)$$

and in fact $a_1(z,\hbar) = a_2(z,0,\hbar) + O(\hbar)$. $\square$

8.1.3 The global definition of $I^k(X,\Lambda)$.

Let (Λ,ψ) be an exact Lagrangian submanifold of T^*X. We can find a locally finite open cover of Λ by open sets Λ_i such that each Λ_i is defined by a generating function ϕ_i relative to a fibration $\pi_i : Z_i \to U_i$ where the U_i are open subsets of X. We let $I_0^k(X,\Lambda)$ consist of those $\frac{1}{2}$-densities which can be written as a finite sum of the form

$$\mu = \sum_{j=1}^{N} \mu_{i_j}, \qquad \mu_{i_j} \in I_0^k(X,\Lambda_{i_j}).$$

By the results of the preceding section we know that this definition is independent of the choice of open cover and of the local descriptions by generating functions.

We then define the space $I^k(X,\Lambda)$ to consist of those $\frac{1}{2}$-densities μ on X such that $\rho\mu \in I_0^k(X,\Lambda)$ for every C^∞ function ρ on X of compact support.

8.2 Semi-classical Fourier integral operators.

Let X_1 and X_2 be manifolds, let

$$X = X_1 \times X_2$$

and let

$$M_i = T^*X_i, \qquad i = 1,2.$$

Finally, let (Γ,Ψ) be an exact canonical relation from M_1 to M_2 so

$$\Gamma \subset M_1^- \times M_2.$$

Let

$$\varsigma_1 : M_1^- \to M_1, \qquad \varsigma_1(x_1,\xi_1) = (x_1,-\xi_1)$$

so that

$$\Lambda := (\varsigma_1 \times \mathrm{id})(\Gamma)$$

and
$$\psi = \Psi \circ (\varsigma_1 \times \mathrm{id})$$
gives an exact Lagrangian submanifold (Λ, ψ) of
$$T^*X = T^*X_1 \times T^*X_2.$$
Associated with (Λ, ψ) we have the space of compactly supported oscillatory $\frac{1}{2}$-densities $I_0^k(X, \Lambda)$. Choose a nowhere vanishing density on X_1 which we will denote (with some abuse of language) as dx_1 and similarly choose a nowhere vanishing density dx_2 on X_2. We can then write a typical element μ of $I_0^k(X, \Lambda)$ as
$$\mu = u(x_1, x_2, \hbar) dx_1^{\frac{1}{2}} dx_2^{\frac{1}{2}}$$
where u is a smooth function of compact support in all three "variables".

Recall that $L^2(X_i)$ is the intrinsic Hilbert space of L^2 half densities on X_i. Since u is compactly supported, we can define the integral operator
$$F_\mu = F_{\mu,\hbar} : \quad \text{Ł}^2(X_1) \to L^2(X_2)$$
by
$$F_\mu(f dx_1^{\frac{1}{2}}) = \left(\int_{X_1} f(x_1) u(x_1, x_2) dx_1 \right) dx_2^{\frac{1}{2}}. \quad (8.3)$$
We will denote the space of such operators by
$$\mathcal{F}_0^k(\Gamma)$$
and call them compactly supported **semi-classical Fourier integral operators**. We could, more generally, demand merely that $u(x_1, x_2, \hbar) dx_1^{\frac{1}{2}}$ be an element of $L^2(X_1)$ in this definition, in which case we would drop the subscript 0.

Let X_1, X_2 and X_3 be manifolds, let $M_i = T^*X_i,\ i = 1, 2, 3$ and let
$$(\Gamma_1, \Psi_1) \in \mathrm{Morph}_{exact}(M_1, M_2), \quad (\Gamma_2, \Psi_2) \in \mathrm{Morph}_{exact}(M_2, M_3)$$
be exact canonical relations. Let
$$F_1 \in \mathcal{F}_0^{m_1}(\Gamma_1) \quad \text{and} \quad F_2 \in \mathcal{F}_0^{m_2}(\Gamma_2).$$
Finally, let
$$n = \dim X_2.$$

Theorem 37 *If Γ_2 and Γ_1 are transversally composible, then*

$$F_2 \circ F_1 \in \mathcal{F}_0^{m_1+m_2+\frac{n}{2}}((\Gamma_2, \Psi_2) \circ (\Gamma_1, \psi_1). \tag{8.4}$$

where the composition of exact canonical relations is given in(4.56) and(4.57).

Proof. By partition of unity we may assume that we have fibrations

$$\pi_1 : X_1 \times X_2 \times S_1 \to X_1 \times X_2, \quad \pi_2 : X_2 \times X_3 \times S_2 \to X_2 \times X_3$$

where S_1 and S_2 are open subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ and that ϕ_1 and ϕ_2 are generating functions for Γ_1 and Γ_2 with respect to these fibrations. We also fix nowhere vanishing $\frac{1}{2}$-densities $dx_i^{\frac{1}{2}}$ on X_i, $i = 1, 2, 3$. So F_1 is an integral operator with respect to a kernel of the form (8.3) where

$$u_1(x_1, x_2, \hbar) = \hbar^{m_1 - \frac{d_1}{2}} \int a_1(x_1, x_2, s_1, \hbar) e^{i\frac{\phi_1(x_1, x_2, s_1)}{\hbar}} ds_1$$

and F_2 has a similar expression (under the change $1 \mapsto 2$, $2 \mapsto 3$). So their composition is the integral operator

$$f dx_1^{\frac{1}{2}} \mapsto \left(\int_{X_1} f(x_1) u(x_1, x_3, \hbar) dx_1 \right) dx_3^{\frac{1}{2}}$$

where

$$u(x_1, x_3, \hbar) = \hbar^{m_1 + m_2 - \frac{d_1 + d_2}{2}} \times$$

$$\int a_1(x_1, x_2, s_1, \hbar) a_2(x_2, x_3, s_2, \hbar) e^{i\frac{\phi_1 + \phi_2}{\hbar}} ds_1 ds_2 dx_2.$$

By Theorem 30 $\phi_1(x_1, x_2, s_1) + \phi_2(x_2, x_3, s_2)$ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration

$$X_1 \times X_3 \times (X_2 \times S_1 \times S_2) \to X_1 \times X_3,$$

and by (4.57) this is a generating function for $(\Gamma_2, \Psi_2) \circ (\Gamma_1, \Psi_1)$. Since the fiber dimension is $d_1 + d_2 + n$ and the exponent of $\hbar$ in the above expression is $m_1 + m_2 - \frac{d_1 + d_2}{2}$ we obtain (8.4). □

8.3 The symbol of an element of $I^k(X,\Lambda)$.

Let Λ be a Lagrangian submanifold of T^*X. We have attached to Λ the space $I^k(X,\Lambda)$ of oscillating $\frac{1}{2}$-densities. The goal of this section is to give an intrinsic description of the quotient

$$I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$$

as sections of line bundle $\mathbb{L} \to \Lambda$. This line bundle will locally look like the line bundle $|T\Lambda|^{\frac{1}{2}}$ whose sections are $\frac{1}{2}$-densities on Λ. However we will have to tensor this bundle with the Maslov bundle in order to get a precise global description of $I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$. This was a key idea of Keller and Maslov.

Victor: Citations to Keller and Maslov ?

8.3.1 A local description of $I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$.

Let S be an open subset of $\mathbb{R}^d$ and suppose that we have a generating function $\phi = \phi(x,s)$ for Λ with respect to the fibration

$$X \times S \to X, \qquad (x,s) \mapsto x.$$

Fix a C^∞ nowhere vanishing $\frac{1}{2}$-density ν on X so that any other smooth $\frac{1}{2}$-density μ on X can be written as

$$\mu = u\nu$$

where u is a C^∞ function on X.

The critical set C_ϕ is defined by the d independent equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1,\dots,d \tag{8.5}$$

The fact that ϕ is a generating function of Λ asserts that the map

$$\lambda_\phi : C_\phi \to T^*X, \qquad (x,s) \mapsto (x, d\phi_X(x,s)) \tag{8.6}$$

is a diffeomorphism of C_ϕ with Λ. To say that $\mu = u\nu$ belongs to $I_0^k(X,\Lambda)$ means that the function $u(x,\hbar)$ can be expressed as the oscillatory integral

$$u(x,\hbar) = \hbar^{k-\frac{d}{2}} \int a(x,s,\hbar) e^{i\frac{\phi(x,s)}{\hbar}} ds, \ \text{where} \quad a \in C_0^\infty(X \times S \times \mathbb{R}). \tag{8.7}$$

Proposition 13 *If* $a(x,s,0) = 0$ *on* C_ϕ *then* $\mu \in I_0^{k+1}(X,\Lambda)$.

Proof. If $a(x,s,0)=0$ on C_ϕ then by the description (8.5) of C_ϕ we see that we can write

$$a = \sum_{j=1}^d a_j(x,s,\hbar)\frac{\partial \phi}{\partial s_j} + a_0(x,s,\hbar)\hbar.$$

We can then write the integral (8.7) as $v + u_0$ where

$$u_0(x,\hbar) = \hbar^{k+1-\frac{d}{2}} \int a_0(x,s,\hbar) e^{i\frac{\phi(x,s)}{\hbar}} ds$$

so

$$\mu_0 = u_0\nu \in I_0^{k+1}(X,\Lambda)$$

and

$$\begin{aligned} v &= \hbar^{k-\frac{d}{2}} \sum_{j=1}^d \int a_j(x,s,\hbar)\frac{\partial\phi}{\partial s_j} e^{i\frac{\phi}{\hbar}} ds \\ &= -i\hbar^{k+1-\frac{d}{2}} \sum_{j=1}^d \int a_j(x,s,\hbar)\frac{\partial}{\partial s_j} e^{i\frac{\phi}{\hbar}} ds \\ &= i\hbar^{k+1-\frac{d}{2}} \sum_{j=1}^d \int \left(\frac{\partial}{\partial s_j} a_j(x,s,\hbar)\right) e^{i\frac{\phi}{\hbar}} ds \end{aligned}$$

so

$$v = i\hbar^{k+1-\frac{d}{2}} \int b(x,s,\hbar) e^{i\frac{\phi}{\hbar}} ds \quad \text{where} \quad b = \sum_{j=1}^d \frac{\partial a_j}{\partial s_j}. \tag{8.8}$$

This completes the proof of Proposition 13. □

This proof can be applied inductively to conclude the following sharper result:

Proposition 14 *Suppose that* $\mu = u\nu \in I_0^k(X,\Lambda)$ *where* u *is given by (8.7) and for* $i = 0,\ldots,\ell$

$$\frac{\partial^i a}{\partial \hbar^i}(x,s,0)$$

vanishes on C_ϕ. *Then*

$$\mu \in I_0^{k+2\ell+1}(X,\Lambda).$$

As a corollary we obtain:

Proposition 15 *If a vanishes to infinite order on C_ϕ then $\mu \in I^\infty(X, \Lambda)$, i.e.*

$$\mu \in \bigcap_k I^k(X, \Lambda).$$

We will now use stationary phase to prove the following converse to Proposition 13:

Proposition 16 *If $\mu \in I_0^{k+1}(X, \Lambda)$ then the restriction of $a(x, s, 0)$ to C_ϕ vanishes identically.*

Recall the following fact from the formula of stationary phase: Suppose that Y is a manifold with a nowhere vanishing density dy and that $\psi : Y \to \mathbb{R}$ is a a C^∞ function on Y with a single non-degenerate critical point p_0. Suppose that $f \in C_0^\infty(Y)$. The formula of stationary phase (see Chapter **??**) implies that

$$I(\hbar) := \int_Y f(y) e^{i\frac{\psi(y)}{\hbar}} dy$$

satisfies

$$I(\hbar) = \hbar^{\frac{\dim Y}{2}} \left(\gamma f(p_0) + O(\hbar)\right)$$

where γ is a non-zero constant. In particular, if we write $m = \dim Y$

$$I(\hbar) = O(\hbar^{\frac{m}{2}+1}) \quad \Leftrightarrow \quad f(p_0) = 0. \tag{8.9}$$

Proof of Proposition 16. As usual, we choose a nowhere vanishing $\frac{1}{2}$-density on X and write $\mu = u\nu$ where

$$u(x, \hbar) = \hbar^{k-\frac{d}{2}} \int a(x, s, \hbar) e^{i\frac{\phi(x,s)}{\hbar}} ds$$

where d is the fiber dimension. Let $p_0 = (x_0, s_0) \in C_\phi$ and let

$$(x_0, \xi_0) = \lambda_\phi(p_0) \in \Lambda.$$

Let Γ be a Lagrangian submanifold of T^*X which is horizontal and which intersects Λ transversally at (x_0, ξ_0). We will view Γ as a "point" of T^*X, that is as an element of

$$\mathrm{Morph}(\mathrm{pt.}, T^*X).$$

Since Γ is horizontal, it is defined by a generating function $\chi \in C^\infty(X)$. In other words, $(x, \xi) \in \Gamma$

if and only if $d\chi(x)=\xi$. Let b be any element of $C_0^\infty(X)$ with $b(x_0)\neq 0$. Let

$$v(x)=b(x)e^{-i\frac{\chi(x)}{\hbar}}.$$

This is the integral kernel of a semi-classical Fourier integral operator

$$F_v\in I^0(\Gamma^\dagger)$$

associated to the canonical relation

$$\Gamma^\dagger\in\mathrm{Morph}(T^*X,\mathrm{pt.}).$$

Since

$$\Gamma^\dagger \pitchfork \Lambda$$

we can compose F_v with $\mu=udx^{\frac{1}{2}}\in I_0^{k+1}(X,\Lambda)$ to get an element

$$\int_X v(x,\hbar)u(x,\hbar)dx\in I^{k+1+\frac{n}{2}}(\mathrm{pt.}).$$

This says that

$$\int_X v(x,\hbar)u(x,\hbar)dx=O(\hbar^{k+1+\frac{n}{2}}).$$

So

$$\hbar^{k-\frac{d}{2}}\int b(x)a(x,s,\hbar)e^{i\frac{-\chi(x)+\phi(x,s)}{\hbar}}dxds=O(\hbar^{k+1+\frac{n}{2}}).$$

So if we set

$$\psi(x,s)=-\chi(x)+\phi(x,s)$$

then

$$\int b(x)a(x,s,0)e^{i\frac{\psi(x,s)}{\hbar}}dxds=O(\hbar^{\frac{d+n}{2}+1}).$$

We want to apply (8.9) with $Y=X\times S$ and $f=ba$. First observe that (x_0,s_0) is a critical point of ψ. Indeed

$$\frac{\partial\psi}{\partial s_i}=\frac{\partial\phi}{\partial s_i}=0$$

because $(x_0,s_0)\in C_\phi$ and

$$d_X\psi(x_0)=-d\chi(x_0)+d_X\phi(x_0,s_0)=-\xi_0+\xi_0=0.$$

We claim that (x_0, s_0) is a non-degenerate critical point of ψ. Indeed, we know that $\psi(x,s) = -\chi(x) + \phi(x,s)$ is a generating function for pt. $= \Gamma^\dagger \circ \Lambda$ with respect to the fibration $X \times S \to$ pt.. The condition for being such a generating function says that the differentials of all the partial derivatives of ψ be linearly independent at (x_0, s_0) which is the same as saying that (x_0, s_0) is a non-degenerate critical point. So $b(x_0)a(x_0, s_0, 0) = 0$ and since $b(x_0) \neq 0$ we must have $a(x_0, s_0, 0) = 0$. Since this is true at all points of C_ϕ we conclude that $a(x, s, 0) \equiv 0$ on C_ϕ. $\square$

We can now summarize the results of the last few propositions: Given $\mu \in I_0^k(X, \Lambda)$, suppose that we can write $\mu = u dx^{\frac{1}{2}}$ where $dx^{\frac{1}{2}}$ is a nowhere vanishing $\frac{1}{2}$-density on X and suppose there is a generating function ϕ for Λ valid over an open set containing the support of μ such that u is of the form

$$u = \hbar^{k-\frac{d}{2}} \int a(x, s, \hbar) e^{i\frac{\phi(x,s)}{\hbar}} ds$$

where $a \in C_0^\infty(X \times S \times \mathbb{R})$. We know from Proposition 13 that the function $a(x, s, 0)_{|C_\phi}$ depends only on the equivalence class of μ mod $I_0^{k+1}(X, \Lambda)$ (once ϕ is fixed) and from Propositions 16 and 13 that the map

$$\mu \mapsto a(x, s, 0)_{|C_\phi}$$

is an isomorphism of $I_0^k(X, \Lambda)/I_0^{k+1}(X, \Lambda)$ with $C_0^\infty(C_\phi)$. Now the map

$$\lambda_\phi : C_\phi \to \Lambda; \qquad (x, s) \mapsto (x, d\phi_X(x, s))$$

is a diffeomorphism. So we have proved:

Theorem 38 *Let (Λ, ψ) be an exact Lagrangian submanifold of T^*X and ϕ a generating function for (Λ, ψ) relative to $\pi : X \times S \to X$ and let ν be a nowhere vanishing $\frac{1}{2}$-density on X so that every element $\mu = u\nu$ of $I_0^k(X, \Lambda)$ has a representation as an oscillatory integral of the form (8.7). For each $\mu \in I_0^k(X, \Lambda)$ define the symbol*

$$\sigma_\phi(\mu) \in C_0^\infty(\Lambda)$$

by

$$\sigma_\phi(\mu)(x, \xi) = a(x, s, 0) \textit{ where } (x, s) \in C_\phi \textit{ and } \lambda_\phi(x, s) = (x, \xi) \tag{8.10}$$

for every $(x,\xi)\in\Lambda$. Then σ_ϕ defines an isomorphism

$$\sigma_\phi:\quad I_0^k(X,\Lambda)/I_0^{k+1}(X,\Lambda)\cong C_0^\infty(\Lambda).$$

The isomorphism σ_ϕ depends on the choice of the generating function ϕ. We shall remedy this by reinterpreting $\sigma_\phi(\mu)$ as a section of an appropriate line bundle. Recall from Section 5.13 that the generating function ϕ gives a local flat trivialization of the line bundle $\mathbb{L}_{\text{Maslov}}$. We shall show in the next section that if we use these trivializations and our choice of $\frac{1}{2}$-densities to identify $\sigma_\phi(\mu)$ as a section of

$$|T\Lambda|^{\frac{1}{2}}\otimes\mathbb{L}_{\text{Maslov}}$$

then the resulting section is independent of all these choices and we will be able to define an isomorphism of $I_0^k(X,\Lambda)/I_0^{k+1}(X,\Lambda)$ with smooth sections of compact support of $|T\Lambda|^{\frac{1}{2}}\otimes\mathbb{L}_{\text{Maslov}}$.

8.3.2 The global definition of the symbol.

Rewriting the local definition of $I_0^k(X,\Lambda)$.

Let $\pi:Z\to X$ be an enhanced fibration. This means that the fibers are equipped with a $\frac{1}{2}$-density and hence that the corresponding canonical relation

$$\Gamma_\pi\in\text{Morph}(T^*Z,T^*X),\quad \Gamma_\pi=H^*(Z)$$

is equipped with a $\frac{1}{2}$-density. Recall that this defines a pushforward map on $\frac{1}{2}$-densities of compact support:

$$\pi_*C_0^\infty(|Z|^{\frac{1}{2}})\to C_0^\infty(|X|^{\frac{1}{2}}).$$

Let $\upsilon=\upsilon(z,\hbar)$ be a smooth $\frac{1}{2}$-density of compact support on Z depending smoothly on $\hbar$. Then we can rewrite (8.1) as

$$\mu=\hbar^{k-\frac{d}{2}}\pi_*\left(\upsilon e^{i\frac{\phi}{\hbar}}\right). \tag{8.11}$$

By definition, an element of $I_0^k(X,\Lambda)$ is a $\frac{1}{2}$-density on X which can be written as a finite sum of such terms.

Rewriting the local definition of the symbol.

Recall that ϕ defines the horizontal Lagrangian submanifold $\Lambda_\phi \subset T^*Z$, and so a diffeomorphism

$$\gamma_\phi : Z \to \Lambda_\phi, \quad z \mapsto (z, d\phi(z))$$

and hence a pushforward isomorphism

$$\gamma_{\phi *} : \ C_0^\infty(|Z|^{\frac{1}{2}}) \to C_0^\infty(|\Lambda_\phi|^{\frac{1}{2}}).$$

By assumption,

$$\Gamma_\pi \pitchfork \Lambda_\phi$$

and, locally,

$$\Lambda = \Gamma_\pi(\Lambda_\phi).$$

The enhancement of Γ_π defines a map

$$\Gamma_{\pi *} : C_0^\infty(|\Lambda_\phi|^{\frac{1}{2}}) \to C_0^\infty(|\Lambda|^{\frac{1}{2}}).$$

Hence

$$\Gamma_{\pi *} \circ \gamma_{\phi *} : \ C_0^\infty(|Z|^{\frac{1}{2}}) \to C_0^\infty(\Lambda).$$

We now define

$$\sigma_{\phi,\mathrm{new}}(\mu) := (2\pi)^{-\frac{d}{2}} \hbar^k e^{\frac{\pi i}{4}\operatorname{sgn}_\phi} (\Gamma_{\pi *} \circ \gamma_{\phi *}) \left(\upsilon(z,0) e^{i\frac{\phi(z)}{\hbar}} \right) \tag{8.12}$$

where

$$d = \dim Z - \dim X$$

and where sgn_ϕ is defined in Section 5.13.

Relation of the new local definition to the old one.

Let us see how this new definition of the symbol is related to the one given in Theorem 38. We begin by being more explicit about the map $\Gamma_{\pi *}$. Let

$$M = T^*Z.$$

The fact that $\Gamma_\pi \pitchfork \Lambda_\phi$ says that at every $z \in C_\phi$ we have the exact sequence

$$0 \to T_z(C_\phi) \to T_q(\Lambda_\phi) \oplus T_q(\Gamma_\pi) \to T_q M \to 0 \tag{8.13}$$

where $q = \gamma_\phi(z)$. Since M is a symplectic manifold, it carries a canonical $\frac{1}{2}$-density. The enhancement of Γ_π means that Γ_π is equipped with a $\frac{1}{2}$-density, call it τ. If we are given a C^∞ $\frac{1}{2}$-density ρ on Λ_ϕ, the above exact sequence implies that from the $\frac{1}{2}$-density

$$\rho_q \otimes \tau_q \tag{8.14}$$

we get a $\frac{1}{2}$-density, call it $\rho_z^\sharp$ on $T_z(C_\phi)$. So we get a $\frac{1}{2}$-density $\rho^\sharp$ on C_ϕ. Then

$$\Gamma_{\pi *}\rho = (\lambda_\phi^{-1})^* \rho^\sharp \in C^\infty(|\Lambda|^{\frac{1}{2}}). \tag{8.15}$$

Fix a nowhere vanishing $\frac{1}{2}$-density τ_Z on Z and write

$$\upsilon(z,\hbar) = a(z,\hbar)\tau_Z, \qquad a \in C_0^\infty(Z \times \mathbb{R}).$$

Define the function $a^\sharp$ on C_ϕ by

$$a^\sharp(z) = a(z,0), \qquad z \in C_\phi$$

and define the function $\phi^\sharp$ on C_ϕ by

$$\phi^\sharp = \phi_{|C_\phi}.$$

Thus we can write the $\sigma_\phi(\mu)$ as given in equation (8.10) as

$$\sigma_\phi(\mu) = (\lambda_\phi^{-1})^* a^\sharp.$$

Let ψ be the function on Λ defined by

$$\psi := (\lambda_\phi^{-1})^* \phi^\sharp. \tag{8.16}$$

Then it follows directly from these definitions that

$$\sigma_{\phi,\text{new}}(\mu) = \sigma_\phi(\mu)\kappa e^{i\left(\frac{\psi}{\hbar} + \frac{\pi}{4}\,\text{sgn}_\phi\right)} \tag{8.17}$$

where

$$\kappa := (2\pi)^{-\frac{d}{2}}\hbar^k \Gamma_{\pi *}(\gamma_{\phi *}\tau_Z) \tag{8.18}$$

does not depend on μ.

From the above discussion it follows that

Proposition 17 *$\sigma_{\phi,\text{new}}$ depends only on Γ_π but not on its enhancement.*

Proof. Indeed, if we replace τ by $f\tau$ where f is a nowhere vanishing function, then $\pi_*\beta$ is replaced by $\pi_*(f\beta)$ for any $\frac{1}{2}$-density β on Z. This means that in the description (8.11) of μ we must replace υ by $f^{-1}\upsilon$. So in (8.14), we replace τ by $f\tau$ and ρ by $f^{-1}\rho$. So these two changes cancel one another in in (8.14) and hence in (8.12). $\square$

Recall: Fixing the arbitrary constant in the phase function for an exact Lagrangian submanifold.

Recall that if ϕ is a local description of Λ relative to a fibration π, then so is $\phi + c$ where c is a constant.We have fixed this constant using (4.58), so that (8.16) holds in any local description (Z, π, ϕ) of Λ. Then in the factor

$$e^{i\frac{\psi}{\hbar}}$$

occurring in (8.17), the function ψ is precisely the function chosen (in advance of all local parametrizations) on Λ so as to satisfy

$$d\psi = \alpha_\Lambda. \tag{8.19}$$

The factor $e^{\frac{\pi i}{4}\operatorname{sgn}_\phi}$ is a flat section of $\mathbb{L}_{\text{Maslov}}$ and since κ as given by (8.18) is a $\frac{1}{2}$-density on Λ, this allows us to interpret $\sigma_{\phi,\text{new}}(\mu)$ as a section of the line bundle

$$|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\text{Maslov}}.$$

The global definition.

Theorem 39 *The definition of the map*

$$\sigma_{\phi,\text{new}} : I_0^k \to C_0^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\text{Maslov}})$$

is independent of the choice of generating function and fibration and hence defines (locally) an isomorphism

$$\sigma : I^k(X, \Lambda)/I^{k+1}(X, \Lambda) \to C^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\text{Maslov}}).$$

Proof. The second assertion follows from what we proved in the preceding section. So we need to prove the first assertion. By Section 5.12, and the fact that we have fixed the arbitrary constant in all phase functions, we need to prove independence under two kinds of moves - equivalence and increasing the number of fiber variables.

Invariance under equivalence.

So we have (Z_1, π_1, ϕ_1) and (Z_2, π_2, ϕ_2) and a diffeomorphism

$$g : Z_1 \to Z_2$$

with

$$\pi_1 = \pi_2 \circ g \quad \text{and} \quad \phi_1 = \phi_2 \circ g.$$

Then g determines a symplectomorphsim

$$\Gamma_g \in \mathrm{Morph}(T^*Z_1, T^*Z_2)$$

with

$$\Gamma_{\pi_1} = \Gamma_{\pi_2} \circ \Gamma_g \quad \text{and} \quad \gamma_{\phi_2} = \Gamma_g \circ \gamma_{\phi_1}.$$

We may choose the nowhere vanishing $\frac{1}{2}$-densities on Z_1 and Z_2 to be consisitent as we did in Section8.1.2 By Proposition 17 we may also choose the enhancements consistently in the sense that

$$(\pi_1)_* = (\pi_2)_* \circ g_*.$$

We also know that the signatures entering into formula (8.17) are the same for ϕ_1 and ϕ_2. Thus (8.17) gives the same answer for ϕ_1 and ϕ_2.

Invariance under increasing the number of fiber variables.

So now

$$Z_2 = Z_1 \times \mathbb{R}^m$$

and

$$\phi_2(z,y) = \phi_1(z_1) + \frac{1}{2}\langle Ay, y\rangle$$

where A is a non-degenerate symmetric matrix and

$$\pi_2 = \pi_1 \circ \pi, \qquad \pi(z_1, y) = z_1.$$

We choose an enhancement r of $\pi_1 : Z_1 \to X$ and then pick the enhancement of $\pi_2 : Z_2 \to X$ to be $r \otimes dy^{\frac{1}{2}}$. This is legitimate by Proposition 17. So if we choose $dy^{\frac{1}{2}}$ to be the enhancement of π we have

$$\pi_{2*} = \pi_{1*} \circ \pi_* \tag{8.20}$$

as maps from $C_0^\infty(|Z_2|^{\frac{1}{2}}) \to C_0^\infty(|X|^{\frac{1}{2}})$ and

$$\Gamma_{\pi_2 *} = \Gamma_{\pi_1 *} \circ \Gamma_{\pi *} \tag{8.21}$$

as maps from $\frac{1}{2}$-densities on Λ_{ϕ_2} to $\frac{1}{2}$-densities on Λ.

Let us also choose a nowhere vanishing τ_{Z_1} on Z_1 and choose the nowhere vanishing $\frac{1}{2}$-density on Z_2 to be

$$\tau_{Z_2} = \tau_{Z_1} \otimes dy^{\frac{1}{2}}.$$

Let us now rewrite the definitions (8.1), (8.10) and (8.12) in terms of a general fibration $\pi : Z \to X$ and generating function ϕ as follows: First consider the manifold Z relative to the trivial fibration over itself, and the Lagrangian submanifold $\Lambda_\phi \subset T^*Z$ given by the the function ϕ so that $\Lambda_\phi = \gamma_\phi(Z)$. Let$\tau_Z$ be a nowhere vanishing $\frac{1}{2}$-density on Z. Definition (8.1) (relative to the trivial fibration of Z over itself) says that $I_0^{k-\frac{d}{2}}(Z,\Lambda_\phi)$ consists of all $\frac{1}{2}$ densities on Z of the form

$$\upsilon = \hbar^{k-\frac{d}{2}} a(z,\hbar)\tau_Z e^{i\frac{\phi(z)}{\hbar}}.$$

We may write

$$\upsilon = \upsilon_0 + O(\hbar^{k-\frac{d}{2}+1})$$

where

$$\upsilon_0 = \hbar^{k-\frac{d}{2}} a(z,0)\tau_Z e^{i\frac{\phi(z)}{\hbar}}.$$

The definition of the symbol for this trivial fibration then says that

$$\sigma_{\mathrm{new}}(\upsilon) = \gamma_{\phi_*}\upsilon.$$

If we set

$$\sigma_{\Lambda_\phi} := \gamma_{\phi *}\sigma_Z$$

and use the above representation of υ then

$$\sigma_{\mathrm{new}}(\upsilon) = \gamma_{\phi_*}\upsilon_0 = \hbar^{k-\frac{d}{2}}(\gamma_\phi^{-1})^*\left(a(z,0)e^{i\frac{\phi}{\hbar}}\right)\sigma_{\Lambda_\phi}. \tag{8.22}$$

Now (8.1) says that a general element of $I^k(X,\Lambda)$ can be written locally as

$$\mu = \pi_*\upsilon, \qquad \upsilon \in I^{k-\frac{d}{2}}(Z,\Lambda_\phi)$$

and then (8.12) says that

$$\sigma_{\phi,\mathrm{new}} = e^{i\frac{\pi}{4}\sigma_\phi}\left(\frac{\hbar}{2\pi}\right)^{\frac{d}{2}}\Gamma_{\pi *}\sigma(\upsilon). \tag{8.23}$$

Back to the proof of the theorem: Let $\upsilon_2 \in I_0^{k-\frac{d_2}{2}}(Z_2,\Lambda_{\phi_2})$ and

$$\mu = \pi_{2*}\upsilon_2.$$

Let

$$\upsilon_1 := \pi_*\upsilon_2$$

so that by (8.20) and (8.21)

$$\mu = \pi_{1*}\upsilon_1 = \pi_{1*}(\pi_*\upsilon_2)$$

and

$$\Gamma_{\pi_2 *}\sigma(\upsilon_2) = \Gamma_{\pi_1 *}\left(\Gamma_{\pi *}\sigma(\upsilon_2)\right).$$

So to prove that the two definitions of $\sigma_{\rm new}(\mu)$ coincide, it is enough to show that the two definitions of $\sigma(\upsilon_1)$ - the one associated with the trivial fibration of Z_1 over itself and the generating function ϕ_1, and the one associated the the fibration $\pi : Z_2 \to Z_1$ and ϕ_2 - coincide.

Write

$$\upsilon_2 = \hbar^{k-\frac{d_2}{2}} a(z_1,y,\hbar) e^{i\frac{\phi_2(z,y)}{\hbar}} \tau_{Z_2}$$

so that

$$\upsilon_1 = \hbar^{k-\frac{d_2}{2}} \left(\int a(z_1,y,\hbar) e^{i\frac{\langle Ay,y\rangle}{2\hbar}} dy\right) e^{i\frac{\phi_1}{\hbar}} \tau_{Z_1}.$$

By stationary phase, this last expression is of the form

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}} \left|\det A\right|^{-\frac{1}{2}} a(z_1,0,0) e^{i\frac{\pi}{4}\operatorname{sgn} A} e^{i\frac{\phi_1}{\hbar}} \tau_{Z_1} + O(\hbar^{k-\frac{d_1}{2}+1}).$$

Hence $\sigma_{\rm new}(\upsilon_1)$ computed for the trivial fibration according to (8.22) is

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}} \left|\det A\right|^{-\frac{1}{2}} (\gamma_{\phi_1}^{-1})^* \left(a(z_1,0,0) e^{i\frac{\phi_1}{\hbar}}\right) e^{i\frac{\pi}{4}\operatorname{sgn} A} \gamma_{\phi_1 *}\tau_{Z_1}. \tag{8.24}$$

We now do the computation of the symbol via the pushforward by $\Gamma_{\pi *}$ of a $\frac{1}{2}$-density on Λ_{ϕ_2}. The $\frac{1}{2}$-density in question is

$$\sigma(\upsilon_2) = \hbar^{k-\frac{d_2}{2}} (\gamma_{\phi_2}^{-1})^* \left(a(z_1,y,0) e^{i\frac{\phi_2}{\hbar}}\right) \gamma_{\phi_2 *}(\tau_{Z_1} \otimes dy^{\frac{1}{2}}).$$

We apply (8.23) to the fibration $\pi : Z_2 \to Z_1$ which says that we must use the preceding expression for $\sigma(\upsilon_2)$ in

$$\left(\frac{\hbar}{2\pi}\right)^{\frac{m}{2}} e^{\frac{\pi i}{4}\operatorname{sgn}} \Gamma_{\pi *}\sigma(\upsilon_2).$$

where sgn is the signature of the fibration π and the function

$$Q: \ y \mapsto \frac{1}{2}\langle Ay, y\rangle$$

on the fibers. This signature is just sgn A. So we get for our second computation:

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}} e^{\frac{\pi i}{4}\operatorname{sgn} A}\Gamma_{\pi*}\left[(\gamma_{\phi_2}^{-1})^*\left(a(z_1,y,0)e^{i\frac{\phi_2}{\hbar}}\right)\gamma_{\phi_2*}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right].$$

The critical set C_{ϕ_2} for the fibration π is the set $y=0$. Identifying this set with Z_1, we see that the map

$$\lambda_{\phi_2,\pi} : C_{\phi_2} \to \Lambda_{\phi_1}$$

is just the map

$$\gamma_{\phi_1} :\ Z_1 \to \Lambda_{\phi_1}$$

so our second computation becomes

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}} e^{\frac{\pi i}{4}\operatorname{sgn} A}(\gamma_{\phi_1}^{-1})^*\left(a(z_1,0,0)e^{i\frac{\phi_1}{\hbar}}\right)\Gamma_{\pi*}\left(\gamma_{\phi_2*}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right).$$

If we compare this with (8.24) we see that the proof of the theorem hinges on showing that

$$\Gamma_{\pi*}\left(\gamma_{\phi_2*}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right) = |\det A|^{-\frac{1}{2}}\gamma_{\phi_1*}\tau_{Z_1}. \quad (8.25)$$

Now $Z_2 = Z_1\times\mathbb{R}^m$ and the map γ_{ϕ_2} factors as

$$\gamma_{\phi_2} = \gamma_{\phi_1}\times\gamma_Q$$

where

$$\gamma_{\phi_1} : Z_1 \to \Lambda_{\phi_1}$$

and

$$\gamma_Q : \mathbb{R}^m \to \Lambda_Q. \quad \gamma_Q(y) = (y,\eta), \quad \eta = Ay.$$

Similarly, the map π factors as

$$\pi = \operatorname{id}\times\wp$$

where

$$\wp : \mathbb{R}^m \to \{0\}.$$

So the theorem amounts to showing that

$$\Gamma_{\wp*}\left(\gamma_{Q*}|dy|^{\frac{1}{2}}\right) = |\det A|^{-\frac{1}{2}}.$$

For this let us go back to the exact sequence (8.13) where now $\phi = Q$ so $C_\phi = \{0\}$ is a point. Here $\frac{1}{2}$-density $\gamma_{Q*}|dy|^{\frac{1}{2}}$ assigns the value one to the basis

$$\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_m}, A\frac{\partial}{\partial y_1},\dots,A\frac{\partial}{\partial y_m}\right)$$

of $T_0(\Lambda_Q)$. The Lagrangian submanifold Γ_π consists of the zero section of $T^*(\mathbb{R}^m)$ and the enhancement by $|dy|^{\frac{1}{2}}$ of Γ_π assigns the value one to the basis

$$\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_m},0,\dots,0\right)$$

of $T_0\Gamma_\pi$.

So the tensor product (8.14) assigns the value one to the basis of $T_0(T^*(\mathbb{R}^m))$ obtained by combining these two bases. But the symplectic $\frac{1}{2}$-density assigns the value $|\det A|^{\frac{1}{2}}$ to this combined basis. This proves that $\Gamma_{\wp *}\left(\gamma_{Q*}|dy|^{\frac{1}{2}}\right)=|\det A|^{-\frac{1}{2}}$. □

Whew!

The general definition of the symbol.

Let Λ be an arbitrary Lagrangian submanifold of T^*X. We can cover Λ by open sets U_i each described by a generating function ϕ_i relative to a fibration $\pi_i : Z_i \to U_i$. By definition, if $\mu \in I_0^k(X,\Lambda)$, we can write μ as a finite sum

$$\mu=\sum_{i=1}^N \mu_i, \quad \text{with} \quad \mu_i=\pi_{i*}\upsilon_i, \quad \upsilon_i \in I_0^{k-\frac{d_i}{2}}(Z_i,\Lambda_{\phi_i})$$

where d_i is the fiber dimension of $Z_i \to X_i$. Define

$$\sigma(\mu)\in C_0^\infty(|\Lambda|^{\frac{1}{2}}\otimes \mathbb{L}_{\text{Maslov}})$$

by

$$\sigma(\mu):=\sum_{i=1}^N \sigma(\mu_i). \tag{8.26}$$

From Theorems 38 and 39 we conclude

Theorem 40 *$\sigma(\mu)$ is well defined and independent of the choices that went into (8.26). The map*

$$\sigma:\quad \mu\mapsto\sigma(\mu)$$

induces a bijection

$$I^k(X,\Lambda)/I^{k+1}(X,\Lambda)\cong C^\infty(|\Lambda|^{\frac{1}{2}}\otimes\mathbb{L}_{\text{Maslov}}).$$

8.4 Symbols of semi-classical Fourier integral operators.

Let X_1 and X_2 be manifolds, and

$$\Gamma \in \mathrm{Morph}(T^*X_1, T^*X_2)$$

be a canonical relation. Let

$$\Lambda = (\varsigma_1 \times \mathrm{id})(\Gamma)$$

where $\varsigma(x_1, \xi_1) = (x_1, -\xi_1)$ so that Λ is a Lagrangian submanifold of $T^*(X_1 \times X_2)$. We have associated to Γ the space of compactly supported semi-classical Fourier integral operators

$$\mathcal{F}_0^k(\Gamma)$$

where $F \in \mathcal{F}_0^k(\Gamma)$ is an integral operator with kernel

$$\mu \in I_0^k(X_1 \times X_2, \Lambda).$$

We define the symbol of F to be

$$\sigma(F) = (\varsigma_1 \times \mathrm{id})_* \sigma(\mu)$$

so that

$$\sigma(F) \in C_0^\infty(|\Gamma|^{\frac{1}{2}} \otimes \mathbb{L}_\Gamma)$$

where

$$(\mathbb{L}_\Gamma)_{(x_1,\xi_1,x_2,\xi_2)} = (\mathbb{L}_\Lambda)_{(x_1,-\xi_1,x_2,\xi_2)}.$$

By Theorem 40 we have an isomorphism

$$\mathcal{F}_0^k(\Gamma)/\mathcal{F}_0^{k+1}(\Gamma) \cong C_0^\infty(|\Gamma|^{\frac{1}{2}} \otimes \mathbb{L}_\Gamma). \tag{8.27}$$

Suppose that X_1, X_2 and X_3 are manifolds and that

$$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2) \text{ and } \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$$

are transversally composible. Let $n = \dim X_2$ and

$$F_i \in \mathcal{F}_0^{k_i}(\Gamma), \quad i = 1, 2.$$

By Theorem 37 we know that

$$F_2 \circ F_1 \in \mathcal{F}_0^{m_1+m_2+\frac{n}{2}}(\Gamma_2 \circ \Gamma_1).$$

So if

$$\sigma_i := \sigma(F_i)$$

we may define

$$\sigma_2 \circ \sigma_1 := \sigma(F_2 \circ F_1).$$

By (8.27) we know that this is well defined and hence gives us a composition law

$$C_0^\infty(|\Gamma_1|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_1}) \times C_0^\infty(|\Gamma_2|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_2}) \to C_0^\infty(|\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_2 \circ \Gamma_1}).$$

This modifies our composition formula for $\frac{1}{2}$-densities in the enhanced symplectic category in that it takes the line bundle $\mathbb{L}_\Gamma$ into account.

8.5 Differential operators on oscillatory $\frac{1}{2}$-densities.

Let

$$P : \quad C^\infty(|X|^{\frac{1}{2}}) \to C^\infty(|X|^{\frac{1}{2}})$$

be an m-th order differential operator as discussed in Section 1.3.7. Let $\sigma(P)$ denote the principal symbol of P as discussed there. In particular, $\sigma(P)$ is a function on T^*X.

Let Λ be a Lagrangian submanifold of T^*X, let $\mu \in I^k(X, \Lambda)$ and let $\sigma(\mu)$ denote the symbol of μ as defined in Theorem 40.

Theorem 41 *If $\mu \in I^k(X, \Lambda)$ then*

$$P\mu \in I^{k-m}(X, \Lambda)$$

and

$$\sigma(P\mu) = \hbar^{-m}\sigma(P)_{|\Lambda}\sigma(\mu). \qquad (8.28)$$

Proof. Let (x_0, ξ_0) be a point of Λ and let ϕ be a generating function for Λ near (x_0, ξ_0) relative to a fibration $\pi : Z \to X$. Then μ has the form (8.2) relative to (Z, π, ϕ) and the choices made in Section 8.1.1. We may differentiate under the integral sign and it is clear that applying D^α to $e^{i\frac{\phi}{\hbar}}$ will have a term $(d_X\phi)^\alpha \cdot \hbar^{-|\alpha|}$ with all other terms being of higher order in $\hbar$. This proves the first statement in the theorem. Equation (8.28) then follows from the local expression (8.10) for the symbol. $\square$

We can be more explicit near points $(x_0, \xi_0) \in \Lambda$ where $\xi_0 \neq 0$. According to the result that we proved in Section 5.9, we can find a coordinate patch $(U, x_1, \ldots, x_n)$ about x_0 such that near (x_0, ξ_0), Λ can be described by a generating function

$$\phi(x, \xi) = x \cdot \xi - \rho(\xi), \quad \rho \in C^\infty(\mathbb{R}^n)$$

relative to the fibration

$$U \times \mathbb{R}^n \to U.$$

See equation (5.13) of Section 5.9.

So near (x_0, ξ_0)

$$\Lambda = \{(x, \xi) | x = \frac{\partial \rho}{\partial \xi}\}$$

and $\mu | U$ is of the form

$$\left(\hbar^{k-\frac{n}{2}} \int b(x, \xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi \right) dx^{\frac{1}{2}} \tag{8.29}$$

where $b \in C^\infty$ is supported on a set $|\xi| \leq N$. By Proposition 13 we may replace $b(x, \xi, \hbar)$ by $b(\frac{\partial \rho}{\partial \xi}, \xi, \hbar)$ up to adding a term in $I^{k+1}(X, \Lambda)$. So modulo $I^{k+1}(X, \Lambda)$ we may write μ as

$$\mu = \left(\hbar^{k-\frac{n}{2}} \int b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi \right) dx^{\frac{1}{2}} \tag{8.30}$$

where

$$b_0(\xi, \hbar) = b(\frac{\partial \rho}{\partial \xi}, \xi, \hbar).$$

Since we have chosen the nowhere vanishing $\frac{1}{2}$-density $dx^{\frac{1}{2}}$, we can regard P as a differential operator on functions, and hence by (8.30)

$$\begin{aligned} P\mu &= \left(\hbar^{k-\frac{n}{2}} \int P(x, D) e^{i\frac{x\cdot\xi}{\hbar}} b_0(\xi, \hbar) e^{-i\frac{\rho(\xi)}{\hbar}} d\xi \right) dx^{\frac{1}{2}} \\ &= \left(\hbar^{k-\frac{n}{2}} \int P(x, \xi) b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi \right) dx^{\frac{1}{2}} \end{aligned}$$

where $P(x, \xi)$ is the total symbol of P as defined in Section 1.3.2. So

$$P\mu = \left(\sum_{\ell=1}^{m} \hbar^{k-\ell-\frac{n}{2}} \int p_\ell(x, \xi) b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi \right) dx^{\frac{1}{2}}. \tag{8.31}$$

This proves that $P\mu \in I^{k-m}(X, \Lambda)$ and gives (8.28).

8.6 The transport equations redux.

Let us write H for the principal symbol of P as in Section 1.2.1 and let us assume that

$$H \equiv 0 \quad \text{on } \Lambda$$

as in Sections 1.2.10 and 1.3. Then by (8.28) and Theorem 40, we know that $P\mu \in I^{k-m+1}(X,\Lambda)$. The first main result of this section will be to compute the symbol of $P\mu$ considered as an element of $I^{k-m+1}(X,\Lambda)$. See formula (8.35) below. To prove (8.35) it is enough to prove it on an open dense subset of Λ since the symbol of $P\mu$ (as an element of $I^{k-m+1}(X,\Lambda)$) is a smooth $\frac{1}{2}$-density on Λ. We will assume in this section that Λ has the property that the set of points $(x,\xi) \in \Lambda,\ \xi \neq 0$ is dense in Λ. So it is enough to prove (8.35) at points (x,ξ) where $\xi \neq 0$ near which Λ has a generating function of the form $\phi(x,\xi) = x\cdot\xi - \rho(\xi), \quad \rho \in C^\infty(\mathbb{R}^n)$ as in the preceding section.

Since Λ is defined by the equations

$$x_i = \frac{\partial \rho}{\partial \xi_i}, \qquad i =, \ldots, n,$$

the fact that $H = p_m$ vanishes identically on Λ implies that

$$H = \sum_{i=1}^n q_i(x,\xi)\left(x_i - \frac{\partial \rho}{\partial \xi_i}\right). \tag{8.32}$$

Thus the highest order term in the multiple of $dx^{\frac{1}{2}}$ in (8.31) can be written as

$$\begin{aligned} \hbar^{k-\frac{n}{2}-m}\int p_m(x,\xi)b_0(\xi,\hbar)e^{i\frac{\phi}{\hbar}}d\xi &= \hbar^{k-\frac{n}{2}-m}\sum_j \int q_j(x,\xi)b_0(x,\xi)\frac{\partial \phi}{\partial \xi_j}e^{i\frac{\phi}{\hbar}}d\xi \\ &= \hbar^{k-\frac{n}{2}-m}\sum_j \int q_j b_0 \frac{\hbar}{i}\frac{\partial}{\partial \xi_j}\left(e^{i\frac{\phi}{\hbar}}\right)d\xi \\ &= \hbar^{k-\frac{n}{2}-m+1}\int i\sum_j \frac{\partial}{\partial \xi_j}\left(q_j b_0\right)e^{i\frac{\phi}{\hbar}}d\xi. \end{aligned}$$

So, by (8.31), we may write

$$P\mu = \hbar^{k-\frac{n}{2}-m+1}\left(\int a(\xi,\hbar)e^{i\frac{\phi}{\hbar}}d\xi\right)dx^{\frac{1}{2}} \text{ mod } I^{k-m+2}(X,\Lambda)$$

where

$$a = \iota^* \left(i \sum_j \frac{\partial}{\partial \xi_j} (q_j b_0) + p_{m-1} b_0 \right)$$

and ι denotes the inclusion

$$\iota : \mathbb{R}^n \to U \times \mathbb{R}^n, \qquad \xi \mapsto \left(\frac{\partial \rho}{\partial \xi}, \xi \right).$$

We decompose a into two terms

$$a = a_I + a_{II}$$

where

$$a_I := \iota^* \left(i \sum_j q_j \frac{\partial}{\partial \xi_j} b_0 \right)$$

and

$$a_{II} := \iota^* \left(\left(p_{m-1} + i \sum_j \frac{\partial}{\partial \xi_j} q_j \right) b_0 \right)$$

and will give a geometric interpretation to each of these terms.

We begin with a_I. Since H has the form (8.32),

$$\iota^* \frac{\partial H}{\partial x_j} = q_j \left(\frac{\partial \rho}{\partial \xi}, \xi \right).$$

Let π denote the diffeomorphism

$$\pi : \Lambda \to \mathbb{R}^n, \qquad (x, \xi) \mapsto \xi.$$

Since

$$v_H = \sum_i \left(\frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

is tangent to Λ, we see that the diffeomorphism π maps the restriction of v_H to Λ to

$$\tilde{v} := - \sum_j q_j \left(\frac{\partial \rho}{\partial \xi}, \xi \right) \frac{\partial}{\partial \xi_j}$$

and so

$$a_I = \frac{1}{i} D_{v_H|\Lambda} \pi^* b_0. \tag{8.33}$$

We now turn to a_{II}. Let ν be the $\frac{1}{2}$-density on Λ given by

$$\nu := \pi^* d\xi^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
D_{v_H|\Lambda}\nu &= \pi^*\left(D_{\tilde v} d\xi^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\pi^*\left(\text{div }(\tilde v) d\xi^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\pi^*\left(\text{div }(\tilde v)\right)\nu, \text{and} \\
(\text{div }(\tilde v)) &= \sum_j \left(-\frac{\partial}{\partial \xi_j}\left(q_j(\frac{\partial\rho}{\partial\xi},\xi\right)\right) \\
&= \iota^*\left(-\sum_j \frac{\partial q_j}{\partial \xi_j}(x,\xi) - \sum_{j,\ell}\frac{\partial q_j}{\partial x_\ell}(x,\xi)\frac{\partial^2\rho}{\partial\xi_j\partial\xi_\ell}\right).
\end{aligned}$$

So

$$D_{v_H|\Lambda}\nu = \frac{1}{2}\iota^*\left(-\sum_j \frac{\partial q_j}{\partial \xi_j}(x,\xi) - \sum_{j,\ell}\frac{\partial q_j}{\partial x_\ell}(x,\xi)\frac{\partial^2\rho}{\partial\xi_j\partial\xi_\ell}\right)\nu. \tag{8.34}$$

On the other hand from the formula (8.32) for $H = p_m$ we have

$$\iota^*\sum_\ell \frac{\partial}{\partial x_\ell}\frac{\partial}{\partial \xi_\ell}p_m = \iota^*\left(-\sum_{j,\ell}\frac{\partial q_j}{\partial x_\ell}\frac{\partial^2\rho}{\partial\xi_\ell\partial\xi_j} + \sum_j \frac{\partial q_j}{\partial \xi_j}\right).$$

Multiplying this by $\frac{1}{2}$ and comparing with (8.34) and recalling the formula (1.19) for the sub-principal symbol, we see that

$$a_{II} = \left(\iota^*\sigma_{sub}(P) + \frac{1}{i}\frac{D_{v_H|\lambda}\nu}{\nu}\right)b_0.$$

Hence the symbol of $P\mu$ is given as

$$\sigma(P\mu) = \hbar^{-(m-1)}\left(\frac{1}{i}D_{v_H|\Lambda} + \sigma_{sub}\right)\sigma(\mu). \tag{8.35}$$

We can now go back to the iterative procedure of Chapter 1 for the semi-classical solution of hyperbolic partial differential equations. The first step is to find a Λ on which $H \equiv 0$. The next step is to solve the transport equation

$$\left(\frac{1}{i}D_{v_h|\Lambda} + \sigma_{sub}\right)\sigma(\mu) = 0. \tag{8.36}$$

Along an integral curve $\gamma(t)$ of v_H on λ this reduces to a first order linear ordinary differential equation of the form

$$\frac{d}{dt}\sigma(\mu)(\gamma(t)) + \sigma_{sub}(\gamma(t))\sigma(\mu)(\gamma(t)) = 0$$

with given initial conditions.

Assuming that the integral curves of v_H lying of Λ are well behaved in the sense that they are defined for all t and that there are no periodic or recurrent trajectories, the solution of (8.36) is reduced to the solution of a system of first order linear differential equations.

If we solved (8.36), then we know from Theorem 40 that

$$P\mu \in I^{k-m+2}(X,\Lambda).$$

Let $\sigma_{k+m-2}(\mu)$ now denote the symbol of $P\mu$ considered as an element of $I^{k-m+2}(X,\Lambda)$. We look for a $\nu \in I^{k-1}(X,\Lambda)$ such that

$$P(\mu+\nu) \in I^{k-m+3}(X,\Lambda).$$

For this to be the case $\sigma(\nu)$ must satisfy the inhomogeneous transport equation

$$\hbar^{-m-1}\left(\frac{1}{i}D_{v_h|\Lambda} + \sigma_{sub}\right)\sigma(\nu) = -\sigma_{k+m-2}(\mu). \tag{8.37}$$

This reduces to a system of first order inhomogeneous linear differential equations.

We can now proceed recursively to find $\frac{1}{2}$-densities in $I^k(X,\Lambda)$ such that

$$P\mu \in I^N(X,\Lambda)$$

for arbitrarily large N.

This completes the program outlined in Chapter 1.

8.7 Semi-classical pseudo-differential operators.

These form a special case of the semi-classical Fourier integral operators described in Section 8.2 specialized to the case

$$X_1 = X_2 = X$$

and

$$\Gamma = \mathrm{id} \in \mathrm{Morph}(T^*X, T^*X)$$

so

$$(\varsigma \times \mathrm{id})(\Gamma) = N^*(\Delta)$$

where

$$\Delta \subset X \times X$$

is the diagonal. Clearly Γ is composable with itself so $\mathcal{F}_0(\Gamma)$ is an algebra. If $F_1 \in \mathcal{F}^{k_1}(\Gamma)$ and $F_2 \in \mathcal{F}^{k_2}(\Gamma)$ and either F_1 or F_2 is in $\mathcal{F}_0(\Gamma)$ then their composition is defined and

$$F_2 \circ F_1 \in \mathcal{F}^{k_1+k_2+\frac{n}{2}}(\Gamma)$$

where

$$n = \dim X.$$

So we definitely need to generalize Theorem 37 so as to include clean composition.

In order to avoid the nuisance of accumulating the $\frac{n}{2}$-s we define

$$\Psi^k(X) := \mathcal{F}^{k-\frac{n}{2}}(\Gamma), \qquad \Psi^k_0(X) := \mathcal{F}^{k-\frac{n}{2}}_0(\Gamma). \quad (8.38)$$

Thus if $A_1 \in \Psi^{k_1}(X)$ and $A_2 \in \Psi^{k_2}(X)$ and one or the other is in $\Psi_0(X)$ then

$$A_2 \circ A_1 \in \Psi^{k_1+k_2}(X)$$

We call $\Psi_0(X)$ the algebra of **compactly supported semi-classical pseudo-differential operators** on X. We will now examine the local expression for the composition law in this algebra. So we assume that X is an open convex subset of $\mathbb{R}^n$ and that we have chosen the standard $\frac{1}{2}$ density $dx^{\frac{1}{2}}$ on X. A generating function for $N^*\Delta$ is given as follows: Let

$$\pi : X \times X \times \mathbb{R}^n \to X \times X, \quad (x, y, \xi) \mapsto (x, y).$$

Then according to (5.4) (with a slightly more compact notation)

$$\phi : X \times X \times \mathbb{R}^n \to \mathbb{R}, \quad \phi(x, y, \xi) = (x - y) \cdot \xi$$

is a generating function for $N^*\Delta$.

Dropping the ubiquitous factors of $dx^{\frac{1}{2}}$ we can write $A \in \Psi_0^k(X)$ as being given by the integral kernel

$$A(x_1, x_2, \hbar) = \hbar^{k-\frac{n}{2}} \int a(x_1, x_2, \xi, \hbar) e^{i\frac{(x_1-x_2)\cdot\xi}{\hbar}} d\xi$$

where

$$a \in C_0^\infty(X \times X \times \mathbb{R}^n \times \mathbb{R}).$$

Then

$$(A_1 \circ A_2)(x_1, x_2) = \int A_1(x_1, y, \hbar) A_2(y, x_2) dy$$

so

$$(A_1 \circ A_2)(x_1, x_2) = \hbar^\ell \int a_1(x_1, y, \xi_1, \hbar) a_2(y, x_2, \xi_2, \hbar) e^{i\frac{\xi_1\cdot(x_1-y)+\xi_2\cdot(y-x_2)}{\hbar}} d\xi_1 d\xi_2 dy \tag{8.39}$$

where

$$\ell = k_1 + k_2 - n.$$

Our task is to disentangle this formula.

8.7.1 The right-handed symbol calculus of Kohn and Nirenberg.

Make the changes of coordinates

$$\xi = \xi_1, \ \eta = \xi_1 - \xi_2, \ z = y - x_2$$

so

$$\xi_1 = \xi, \ \xi_2 = \xi - \eta, \ y = z + x_2$$

in (8.39). Thus

$$\xi_1 \cdot (x_1 - y) + \xi_2 \cdot (y - x_2) = \xi \cdot (x_1 - x_2) - \eta \cdot z$$

is the phase function in the new coordinates.

The amplitude in the new coordinates is a_R where

$$a_R(x_1, x_2, \xi, z, \eta, \hbar) := a_1(x_1, z+x_2, \xi, \hbar) a_2(z+x_2, x_2, \xi-\eta, \hbar). \tag{8.40}$$

Thus the right hand side of (8.39) is equal to

$$\hbar^\ell \int e^{i\frac{\xi\cdot(x_1-x_2)}{\hbar}} \left(\int a_R(x_1, x_2, \xi, z, \eta, \hbar) e^{-i\frac{\eta\cdot z}{\hbar}} d\eta dz \right) d\xi. \tag{8.41}$$

We are now going to apply stationary phase to the integral with respect to z and η occurring in (8.41) for fixed x_1, x_2, ξ. This integral is of the form

$$I(\hbar) = \int f(w) e^{i\frac{\langle Aw,w\rangle}{2\hbar}} dw.$$

In the case at hand

$$w = \begin{pmatrix} z \\ \eta \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.$$

The general stationary phase prescription says that an integral of the above form has the asymptotic expansion

$$I(\hbar) \sim \left(\frac{\hbar}{2\pi}\right)^{\frac{d}{2}} \gamma_A \exp\left(-\frac{i\hbar}{2} b(D)\right) f(0)$$

where

$$b(D) = \sum_{ij} b_{ij} D_i D_j. \qquad (b_{ij}) = B = A^{-1},$$

$$\gamma_A = |\det A|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sgn} A}$$

and d is the dimension of the space over which we are integrating. In the case at hand

$$B = A$$

and

$$\operatorname{sgn} A = 0$$

so

$$\gamma_A = 1.$$

Also $d = 2n$. Let us denote the result of applying this stationary phase formula to the a_R of (8.40) by

$$a_1 \star_R a_2.$$

Then we have the formula

$$a_1 \star_R a_2 = \left(\frac{\hbar}{2\pi}\right)^n \sum_k (i\hbar)^k \frac{1}{k!} \left(D_z D_\eta\right)^k a_R \bigg|_{z=\eta=0}. \tag{8.42}$$

Example. Suppose we take $a_1 = a$ and

$$a_2 = a_2(\xi) = \left(\frac{2\pi}{\hbar}\right)^n \rho(\xi)$$

where

$$\rho \in C_0^\infty(\mathbb{R}^n)$$

and $\rho \equiv 1$ on supp(a).

Then

$$a_R = \left(\frac{2\pi}{\hbar}\right)^n a(x_1, z + x_2, \xi, \hbar)\rho(\xi - \eta)$$

so (8.42) gives

$$a_1 \star_R a_2 = \sum_k (i\hbar)^k \frac{1}{k!} (D_z D_\eta)^k \left(a_1(x_1, z + x_2, \xi, \hbar)\rho(\xi - \eta)\right)\Big|_{z=\eta=0}.$$

But since $\rho \equiv 1$ on a neighborhod of supp a, all terms except the first vanish. Hence

$$a \star_R \left(\left(\frac{2\pi}{\hbar}\right)^n \rho\right) = a.$$

The element

$$\left(\frac{2\pi}{\hbar}\right)^n \rho$$

acts as a right identity on all a whose support is contained in the set where $\rho \equiv 1$.

Remark. If at the beginning of this section we had made the change of variables

$$\xi = \xi_2, \ \eta = \xi_2 - \xi_1, \ z = y - x_1$$

we would obtain an alternative symbol calculus, the "left handed calculus". The same argument will then show that $\left(\frac{2\pi}{\hbar}\right)^n \rho$ is a left identity on all a whose support is contained in the set where $\rho \equiv 1$.

8.8 $I(X, \Lambda)$ as a module over $\Psi_0(X)$.

Let X be a manifold and Λ a Lagrangian submanifold of T^*X. Since semi-classical pseudo-differential operators are special kinds of semi-classical Fourier integral operators - ones associated with the identity morphism of T^*X - we may apply the results of Section

8.2 to conclude that $I_0(X,\Lambda)$ is a module over $\Psi_0(X)$. More precisely, if $A \in \Psi_0^k(X)$ and $\nu \in I_0^\ell(X,\Lambda)$ then it follows from Theorem 37 and our convention on the exponent in $\Psi(X)$ that $A\nu \in I_0^{k+\ell}(X,\Lambda)$. In this section we will use stationary phase once again to obtain a local description of this module structure.

We may assume that X is an open subset of $\mathbb{R}^n$, since we are interested in a local description. For simplicity, we will assume that Λ does not intersect the zero section. So we know from Section 5.9 , see equation (5.13), that locally Λ can be described by the fibration

$$\pi : X \times \mathbb{R}^n \to X, \quad (x,\xi) \mapsto x$$

and a generating function of the form

$$\phi^\sharp(x,\xi) = x\cdot\xi - \phi(\xi).$$

We will work locally where this generating function is valid. So we are assuming that $\nu \in I_0^\ell(X,\Lambda)$ is of the form $f dx^{\frac{1}{2}}$ where

$$f(x,\hbar) = \hbar^{\ell-\frac{n}{2}} \int b(x,\xi,\hbar) e^{i\frac{x\cdot\xi-\phi(\xi)}{\hbar}} d\xi.$$

with

$$b \in C_0^\infty(X \times \mathbb{R}^n \times \mathbb{R}).$$

Let $A \in \Psi^k(X)$, so $= u dx^{\frac{1}{2}} dy^{\frac{1}{2}}$ where

$$u(x,y,\hbar) = \hbar^{k-n} \int a(x,y,\xi,\hbar) e^{i\frac{(x-y)\cdot\xi}{\hbar}} d\xi$$

with $a \in C^\infty(X \times X \times \mathbb{R}^n, \mathbb{R})$ supported in a set

$$\|\xi\| \le C.$$

By definition $Af = g(x,\hbar) dx^{\frac{1}{2}}$ where

$$g(x,\hbar) = \int u(x,y,\hbar) f(y,\hbar) dy$$

and hence is given by $\hbar^{k+\ell-\frac{3}{2}n} \times$ the integral

$$\int a(x,y,\xi,\hbar) b(y,\xi_1,\hbar) e^{i\frac{(x-y)\cdot\xi + y\cdot\xi_1 - \phi(\xi_1)}{\hbar}} d\xi d\xi_1 dy.$$

The amplitude in this integral is

$$a(x,y,\xi,\hbar) b(y,\xi_1,\hbar)$$

and the phase is

$$(x-y)\cdot\xi+y\cdot\xi_1-\phi(\xi_1) = x\cdot\xi-\phi(\xi)+y\cdot(\xi_1-\xi)-(\phi(\xi_1)-\phi(\xi)).$$

Let

$$\phi(\xi_1)-\phi(\xi) = \psi(\xi,\xi_1)\cdot(\xi_1-\xi)$$

so that

$$\psi(\xi,\xi) = \frac{\partial\phi}{\partial\xi}(\xi).$$

Holding x and ξ fixed, make the change of variables

$$\eta := \xi_1-\xi, \qquad z := y-\psi(\xi_1,\xi)$$

so that in the new coordinates the phase is

$$x\cdot\xi-\phi(\xi)+z\cdot\eta.$$

and the amplitude is

$$a^\sharp(x,\xi,\eta,z,\hbar) := a(x,z+\psi(\xi+\eta,\xi),\xi,\hbar)b(z+\psi(\xi+\eta,\xi),\xi+\eta,\hbar).$$

So we have $Af = gdx^{\frac{1}{2}}$ with

$$g = \hbar^m\int b^\sharp(x,\xi,\hbar)e^{i\frac{x\cdot\xi-\phi(\xi)}{\hbar}}d\xi, \quad m = k+\ell-\frac{3n}{2}$$

and

$$b^\sharp(x,\xi,\hbar) = \int a^\sharp(x,\xi,\eta,z,\hbar)e^{-\frac{z\cdot\eta}{\hbar}}dzd\eta.$$

Once again, stationary phase applied to this integral gives

$$b^\sharp(x,\xi,\hbar)\sim\left(\frac{\hbar}{2\pi}\right)^n\exp\left(\frac{i\hbar}{2}D_zD_\eta\right)a^\sharp(x,\xi,\eta,z,\hbar)\Big|_{z=\eta=0}.$$

The leading term in this expansion is

$$a^\sharp(x,\xi,0,0) = a(x,\psi(,\xi,\xi),\xi,\hbar)b(\psi(\xi,\xi),\xi,\hbar) =$$
$$a\left(x,\frac{\partial\phi}{\partial\xi}(\xi),\xi,\hbar\right)b\left(\frac{\partial\phi}{\partial\xi}(\xi),\xi,\hbar\right).$$

Since Λ is the submanifold consisting of all

$$(x,\xi) = \left(\frac{\partial\phi}{\partial\xi}(\xi),\xi,\right)$$

in T^*X we see that the leading term depends only on $b_{|\Lambda}$.

8.9 The trace of a semiclassical Fourier integral operator.

Let X be an n-dimensional manifold, let $M = T^*X$ and let

$$\Gamma : T^*X \twoheadrightarrow T^*X$$

be a canonical relation. Let $\Delta_M \subseteq M \times M$ be the diagonal and let us assume that

$$\Gamma \pitchfork \Delta_M \,.$$

Our goal in this section is to show that if $F \in \mathcal{F}_0^k(\Gamma)$ is a semi-classical Fourier integral operator "quantizing" the canonical relation Γ then one has a trace formula of the form:

$$\operatorname{tr} F = \hbar^{k+n} \sum a_p(h) e^{\frac{i\pi}{\eta_p}} e^{iT_p^*/\hbar} \tag{8.43}$$

summed over $p \in \Gamma \cap \Delta_M$. In this formula n is the dimension of X, the η_p's are Maslov factors, the T_p^* are symplectic invariants of Γ at $p \in \Gamma \circ \Delta_M$ which will be defined below, and $a_p(h) \in C^\infty(\mathbb{R})$.

Let $\varsigma : M \to M$ be the involution, $(x, \xi) \to (x, -\xi)$ and let $\Lambda = \varsigma \circ \Gamma$. We will fix a non-vanishing density, dx, on X and denote by

$$\mu = \mu(x, y, \hbar)\, dx^{\frac{1}{2}}\, dy^{\frac{1}{2}} \tag{8.44}$$

the Schwartz kernel of the operator, F. By definition

$$\mu \in I^k(X \times X, \Lambda)$$

and by (8.46) the trace of F is given by the integral

$$\operatorname{tr} F =: \int \mu(x, x)\, dx \,. \tag{8.45}$$

Here are the details:

Let $\varsigma : M \to M$ be the involution, $(x, \xi) \to (x, -\xi)$ and let $\Lambda = \varsigma \circ \Gamma$. We will fix a non-vanishing density, dx, on X and denote by

$$\mu = \mu(x, y, \hbar)\, dx^{\frac{1}{2}}\, dy^{\frac{1}{2}} \tag{8.46}$$

the Schwartz kernel of the operator, F. By definition

$$\mu \in I^k(X \times X, \Lambda)$$

and by (8.46) the trace of F is given by the integral

$$\operatorname{tr} F =: \int \mu(x,x)\,dx\,. \tag{8.47}$$

We can without loss of generality assume that Λ is defined by a generating function, i.e., that there exists a d-dimensional manifold, S, and a function $\varphi(x,y,s) \in C^\infty(X \times X \times S)$ which generates Λ with respect to the fibration, $X \times X \times S \to X \times X$. Let C_φ be the critical set of φ and $\lambda_\varphi : C_\varphi \to \Lambda$ the diffeomorphism of this set onto Λ. Denoting by $\varphi^\sharp$ the restriction of φ to C_φ and by ψ the function, $\varphi^\sharp \circ \lambda_\varphi^{-1}$, we have by (8.19)

$$d\psi = \alpha_\Lambda \tag{8.48}$$

where α_Λ is the restriction to Λ of the canonical one form, α, on $T^*(X \times X)$.

Lets now compute the trace of F. By assumption μ can be expressed as an oscillatory integral

$$(dx)^{\frac{1}{2}}(dy)^{\frac{1}{2}}\left(h^{k-d/2}\int a(x,y,s,h)e^{\frac{i\varphi(x,y,s)}{\hbar}}\,ds\right)$$

and hence by (8.47)

$$\operatorname{tr} F = \hbar^{k-d/2}\int a(x,y,s,\hbar)e^{i\frac{\varphi(x,y,s)}{\hbar}}\,ds\,dx\,. \tag{8.49}$$

We claim that: *The function*

$$\varphi(x,y,s) : X \times S \to \mathbb{R} \tag{8.50}$$

is a Morse function, and its critical points are in one-one correspondence with the points, $p \in \Gamma \cap \Delta_M$.

Proof. Let Δ_X be the diagonal in $X \times X$ on $\Lambda_\Delta = N^*\Delta_X$ its conormal bundle in $T^*(X \times X) = M \times M$. Then $\varsigma \circ \Lambda_\Delta = \Delta_M$ and hence $\Gamma \pitchfork \Delta_M \Leftrightarrow \Lambda \pitchfork \Lambda_\Delta$. Thus the canonical relations

$$\Lambda : pt \to M \times M$$

and

$$\Lambda_\Delta^t : M \times M \to pt$$

are composable and hence the function (8.50) is a generating function for the Lagrangian manifold "*pt*" with respect to the fibration $X \times S \to pt$. In other words, in more prosaic language, the function (8.50) is a Morse function. Its critical points are the points where

$$\frac{\partial \varphi}{\partial s} = 0$$

and

$$\xi = \frac{\partial \varphi}{\partial x}(x, x, s) = -\frac{\partial \varphi}{\partial y}(x, x, s) = \eta\,;$$

in other words, points $(x, y, s) \in C_\varphi$ with the property $\gamma_\varphi(x, y, s) = (x, \xi, y, \eta)$, $p = (x, \xi) = (y, -\eta)$, hence these points are in one-one correspondence with the points $p \in \Gamma \cap \Delta_M$. $\square$

Since the function (8.50) is a Morse function we can evaluate (8.48) by stationary phase obtaining

$$\operatorname{tr} F = \sum h^{k+\eta} a_p(h) e^{i\frac{\pi}{4}\operatorname{sgn}_p} e^{i\psi(p)/\hbar} \tag{8.51}$$

where sgn_p is the signature of $\varphi(x, x, s)$ at the critical point corresponding to p and

$$\psi(p) = \varphi(x, x, s)\,,$$

the value of $\varphi(x, x, s)$ at this point. This gives us the trace formula (8.43) with $T_p^\sharp = \psi(p)$.

8.9.1 Examples.

Let's now describe how to compute these $T_p^\sharp$'s in some examples: Suppose Γ is the graph of a symplectomorphism

$$f : M \to M\,.$$

Let pr_1 and pr_2 be the projections of $T^*(X \times X) = M \times M$ onto its first and second factors, and let α_X be the canonical one form on T^*X. Then the canonical one form, α, on $T^*(X \times X)$ is

$$(pr_1)^* \alpha_X + (pr_2)^* \alpha_X\,,$$

so if we restrict this one form to Λ and then identify Λ with M via the map, $M \to \Lambda$, $p \to (p, \sigma f(p))$, we get from (8.48)

$$\alpha_X - f^* \alpha_X = d\psi \tag{8.52}$$

and $T_p^\sharp$ is the value of ψ at the point, p.

Let's now consider the Fourier integral operator

$$F^m = \overbrace{F \circ \cdots \circ F}$$

and compute its trace. This operator "quantizes" the symplectomorphism f^m, hence if

$$\text{graph}\, f^m \,\overline{\pitchfork}\, \Delta_M$$

we can compute its trace by (8.43) getting the formula

$$\operatorname{tr} F^m = \hbar^\ell \sum a_{m,p}(\hbar) e^{i\frac{\pi}{4}\sigma_{m,p}} e^{iT^\sharp_{m,p}/\hbar}\,. \tag{8.53}$$

with $\ell = km + (\frac{m-1}{2})n$, the sum now being over the fixed points of f^m. As above, the oscillations, $T^\sharp_{m,p}$, are computed by evaluating at p the function, ψ_m, defined by

$$\alpha_X - (f^m)^*\alpha_X = d\psi_m\,.$$

However,

$$\begin{aligned} \alpha_X - (f^m)^*\alpha_X &= \alpha_X - f^*\alpha_X + \cdots + (f^{m-1})^*\alpha_X - (f^m)^*\alpha\,, \\ &= d(\psi + f^x\psi + \cdots + (f^{m-1})^x\psi) \end{aligned}$$

where ψ is the function (8.48). Thus at $p = f^m(p)$

$$T^\sharp_{m,p} = \sum_{i=1}^{m-1} \psi(p_i)\,, \quad p_i = f^i(p)\,. \tag{8.54}$$

In other words $T^\sharp_{m,p}$ is the sum of ψ over the periodic trajectory $(p_1, \ldots, p_{m-1})$ of the dynamical system

$$f^k\,, \quad -\infty < k < \infty\,.$$

We refer to the next subsection "The period spectrum of a symplectomorphism" for a proof that the $T^\sharp_{m,p}$'s are *intrinsic* symplectic invariants of this dynamical system, i.e., depend only on the symplectic structure of M not on the canonical one form, α_X. (We will also say more about the "geometric" meaning of these $T^\sharp_{m,p}$'s in the next lecture.)

Finally, what about the amplitudes, $a_p(h)$, in formula (8.43)? There are many ways to quantize the symplectomorphism, f, and no canonical way of choosing such a quantization; however, one condition which one can impose on F is that its symbol be of the form:

$$h^{-n}\sigma_\Gamma e^{\frac{i\psi}{\hbar}} e^{i\frac{\pi}{4}\sigma_\varphi}\,, \tag{8.55}$$

in the vicinity of $\Gamma \cap \Delta_M$, where ν_Γ is the $\frac{1}{2}$ density on Γ obtained from the symplectic $\frac{1}{2}$ density, ν_M, on M by the identification, $M \leftrightarrow \Gamma$, $p \to (p, f(p))$. We can then compute the symbol of $a_p(h) \in I^0(pt)$ by pairing the $\frac{1}{2}$ densities, ν_M and ν_Γ at $p \in \Gamma \cap \Delta_M$ as in (7.14) obtaining

$$a_p(0) = |\det(I - df_p)|^{\frac{1}{2}} . \tag{8.56}$$

Remark. The condition (8.55) on the symbol of F can be interpreted as a "unitarity" condition. It says that "microlocally" near the fixed points of f:

$$FF^t = I + O(h) .$$

8.9.2 The period spectrum of a symplectomorphism.

Let (M, ω) be a symplectic manifold. We will assume that the cohomology class of ω is zero; i.e., that ω is exact, and we will also assume that M is connected and that

$$H^1(M, \mathbb{R}) = 0 . \tag{*}$$

Let $f : M \to M$ be a symplectomorphism and let $\omega = d\alpha$. We claim that $\alpha - f^*\alpha$ s exact. Indeed $d\alpha - f^*d\alpha = \omega - f^*\omega = 0$, and hence by (*) $\alpha - f^*\alpha$ s exact. Let

$$\alpha - f^*\alpha = d\psi$$

for $\psi \in C^\infty(M)$. This function is only unique up to an additive constant; however, there are many ways to normalize this constant. For instance if W is a connected subset of the set of fixed points of f, and $j : W \to M$ is the inclusion map, then $f \circ j = j$; so

$$j^*\, d\psi = j^*\alpha - j^*f^*\alpha = 0$$

and hence ψ is constant on W. Thus one can normalize ψ by requiring it to be zero on W.

Example. Let Ω be a smooth convex compact domain in $\mathbb{R}^n$, let X be its boundary, let U be the set of points, (x, ξ), $|\xi| < 1$, in T^*X. If $B : U \to U$ is the billiard map and α the canonical one form on T^*X one can take for $\psi = \psi(x, \xi)$ the function

$$\psi(x, \xi) = |x - y| + C$$

where $(y, n) = B(x, \xi)$. B has no fixed points on U, but it extends continuously to a mapping of $\bar{U}$ on $\bar{U}$ leaving the boundary, W, of U fixed and we can normalize ψ by requiring that $\psi = 0$ on W, i.e., that $\psi(x, \xi) = |x - y|$.

Now let

$$\gamma = p_1, \dots, p_{k+1}$$

be a periodic trajectory of f, i.e.,

$$f(p_i) = p_{i+1} \qquad i = 1, \dots k$$

and $p_{k+1} = p_1$. We define the *period* of γ to be the sum

$$p(\gamma) = \sum_{i=1}^{k} \psi(p_i)\,.$$

Claim: *$P(\gamma)$ is independent of the choice of α and ψ. In other words it is a* symplectic invariant *of f.*

Proof. Suppose $\omega = d\alpha - d\alpha'$. Then $d(\alpha - \alpha') = 0$; so, by (*), $\alpha' - \alpha = dh$ for some function, $h \in C^\infty(M)$. Now suppose $\alpha - f^*\alpha = d\psi$ and $\alpha' - f^*\alpha' = d\psi'$ with $\psi = \psi'$ on the set of fixed points, W. Then

$$d\psi' - d\psi = d(f^*h - h)$$

and since $f^* = 0$ on W

$$\psi' - \psi = f^*h - h\,.$$

Thus

$$\begin{aligned} \sum_{i=1}^{k} \psi'(p_i) - \psi(p_i) &= \sum_{i=1}^{k} h(f(p_i)) - h(p_i) \\ &= \sum_{i=1}^{k} h(p_{i+1}) - h(p_i) \\ &= 0\,. \end{aligned}$$

Hence replacing ψ by ψ' doesn't change the definition of $P(\gamma)$. $\square$

Example Let $p_i = (x_i, \xi_i)$ $i = 1, \dots, k+1$ be a periodic trajectory of the billiard map. Then its period is the sum

$$\sum_{i=1}^{k} |x_{i+1} - x_i|\,,$$

i.e., is the *perimeter* of the polygon with vertices at $x_1, \ldots, x_k$. (It's far from obvious that this is a symplectic invariant of B.)

8.10 The mapping torus of a symplectic mapping.

We'll give below a geometric interpretation of the oscillations, $T^\sharp_{m,p}$, occurring in the trace formula (8.53). First, however, we'll discuss a construction used in dynamical systems to convert "discrete time" dynamical systems to "continuous time" dynamical systems. Let M be a manifold and $f : M \to M$ a diffeomorphism. From f one gets a diffeomorphism

$$g : M \times \mathbb{R} \to M \times \mathbb{R}\,, \quad g(p, q) = (f(p), q + 1)$$

and hence an action

$$\mathbb{Z} \to Diff(M \times \mathbb{R})\,, \quad k \to g^k\,, \tag{8.57}$$

of the group, $\mathbb{Z}$ on $M \times \mathbb{R}$. This action is free and properly discontinuous so the quotient

$$Y = M \times \mathbb{R}/\mathbb{Z}$$

is a smooth manifold. The manifold is called the *mapping torus* of f. Now notice that the translations

$$\tau_t : M \times \mathbb{R} \to M \times \mathbb{R}\,, \, (p, q) \to (p, q + t)\,, \tag{8.58}$$

commute with the action (8.57), and hence induce on Y a one parameter group of translations

$$\tau^\sharp_t : Y \to Y\,,\, -\infty < t < \infty\,. \tag{8.59}$$

Thus the mapping torus construction converts a "discrete time" dynamical system, the "discrete" one-parameter group of diffeomorphisms, $f^k : M \to M$, $-\infty < k < \infty$, into a "continuous time" one parameter group of diffeomorphisms (8.59).

To go back and reconstruct f from the one-parameter group (8.59) we note that the map

$$\iota : M = M \times \{0\} \to M \times \mathbb{R} \to (M \times \mathbb{R})/Z$$

imbeds M into Y as a global cross-section, M_0, of the flow (8.59) and for $p \in M_0$ $\gamma_t(p) \in M_0$ at $t = 1$ and via the identification $M_0 \to M$, the map, $p \to \gamma_1(p)$, is just the map, f. In other words, $f : M \to M$ is the "first return map" associated with the flow (8.59).

We'll now describe how to "symplecticize" this construction. Let $\omega \in \Omega^2(M)$ be an exact symplectic form and $f : M \to M$ a symplectomorphism. For $\alpha \in \Omega^1(M)$ with $d\alpha = \omega$ let

$$\alpha - f^*\alpha = d\varphi \tag{8.60}$$

and lets assume that φ is bounded from below by a positive constant. Let

$$g : M \times \mathbb{R} \to M \times \mathbb{R}$$

be the map

$$g(p, q) = (p, q + \varphi(x))\,. \tag{8.61}$$

As above one gets from g a free properly discontinuous action, $k \to g^k$, of $\mathbb{Z}$ on $M \times \mathbb{R}$ and hence one can form the mapping torus

$$Y = (M \times \mathbb{R})/\mathbb{Z}\,.$$

Moreover, as above, the group of translations,

$$\tau_t : M \times \mathbb{R} \to M \times \mathbb{R}\,,\ \tau_t(p, q) = (p, q + t)\,,$$

commutes with (8.61) and hence induces on Y a one-parameter group of diffeomorphisms

$$\tau_t^\sharp : Y \to Y\,,$$

just as above. We will show, however, that these are not just diffeomorphisms, they are *contacto-morphisms*. To prove this we note that the one-form,

$$\tilde{\alpha} = \alpha + dt\,,$$

on $M \times \mathbb{R}$ is a contact one-form. Moreover,

$$\begin{aligned} g^*\tilde{\alpha} &= f^*\alpha + d(\varphi + t) \\ &= \alpha + (f^*\alpha - \alpha) + d\varphi + dt \\ &= \alpha + dt = \tilde{\alpha} \end{aligned}$$

by (8.60) and

$$(\tau_a)^*\tilde{\alpha} = \alpha + d(t + a) = \alpha + dt = \tilde{\alpha}$$

so the action of $\mathbb{Z}$ on $M \times \mathbb{R}$ and the translation action of $\mathbb{R}$ on $M \times \mathbb{R}$ are both actions by groups of contactomorphisms. Thus, $Y = (M \times \mathbb{R})/\mathbb{Z}$ inherits from $M \times \mathbb{R}$ a contact structure and the one-parameter group of diffeomorphisms, $\tau_t^\sharp$, preserves this contact structure.

Note also that the infinitesimal generator, of the group translations, τ_t, is just the vector field, $\frac{\partial}{\partial t}$, and that this vector field satisfies

$$\iota(\frac{\partial}{\partial t})\tilde{\alpha} = 1$$

and

$$\iota(\frac{\partial}{\partial t})\, d\tilde{\alpha} = 0\,.$$

Thus $\frac{\partial}{\partial t}$ is the contact vector field associated with the contract form $\tilde{\alpha}$, and hence the infinitesimal generator of the one-parameter group, $\tau_t^\sharp : Y \to Y$ is the contact vector field associated with the contract form on Y.

Comments:

1. The construction we've just outlined involves the choice of a one-form, α, on M with $d\alpha = \omega$ and a function, φ, with $\alpha = f^x\alpha = d\varphi$; however, it is easy to see that the contact manifold, Y, and one-parameter group of contactomorphisms are uniquely determined, up to contractomorphism, independent of these choices.

2. Just as in the standard mapping torus construction f can be shown to be "first return map" associated with the one-parameter group, $\tau_t^\sharp$.

We can now state the main result of this section, which gives a geometric description of the oscillations, $T^\sharp_{m,p}$, in the trace formula.

Theorem 42 *The periods of the periodic trajectories of the flow, $\tau_t^\sharp$, $-\infty < t < \infty$, coincide with the "length" spectrum of the symplectomorphism, $f : M \to M$.*

Proof. For $(p, a) \in M \times \mathbb{R}$,

$$g^m(p,a) = (f^m(p), q + \varphi(p) + \varphi(p_1) + \cdots + \varphi(p_{m-1})$$

with $p_i = f^i(p)$. Hence if $p = f^m(p)$

$$g^m(p,a) = \tau_{T^\sharp}(p,a)$$

with

$$T^\sharp = T^\sharp_{m,p} = \sum_{i=1}^m \varphi(p_i)\,, \quad p_i = f^i(p)\,.$$

Thus if q is the projection of (p,a) onto Y the trajectory of $\tau^\sharp$ through q is periodic of period $T^\sharp_{m,p}$. □

Via the mapping torus construction one discovers an interesting connection between the trace formula in the preceding section and a trace formula which we described in Section 7.7.4.

Let β be the contact form on Y and let

$$M^\sharp = \{(y,\eta) \in T^*Y\,,\, \eta = t\beta_y\,,\, t \in \mathbb{R}_+\}\,.$$

It's easy to see that $M^\sharp$ is a symplectic submanifold of T^*Y and hence a symplectic manifold in its own right. Let

$$H : M^\sharp \to \mathbb{R}^+$$

be the function $H(y, tB_y) = t$. Then Y can be identified with the level set, $H = 1$ and the Hamiltonian vector field ν_H restricted to this level set coincides with the contact vector field, ν, on Y. Thus the flow, $\tau^\sharp_t$, is just the Hamiltonian flow, $\exp t\nu_H$, restricted to this level set. Let's now compute the "trace" of $\exp t\nu_H$ as an element in the category $\tilde{S}$ (the enhanced symplectic category).

The computation of this trace is essentially identical with the computation we make at the end of Section 7.7.4 and gives as an answer the union of the Lagrangian manifold

$$\Lambda_{T^\sharp_{m,p}} \subset T^*\mathbb{R}\,,\, m \in \mathbb{Z}\,,$$

where the $T^\sharp$'s are the elements of the period spectrum of ν_H and $\Lambda_{T^\sharp}$ is the cotangent fiber at $t = T$. Moreover, each of these $\Lambda_{T^\sharp}$'s is an element of the enhanced symplectic category, i.e. is equipped with a $\frac{1}{2}$-density $\nu_{T^\sharp_{m,p}}$ which we computed to be

$$\overline{T}^\sharp_{m,p}|I - df^m_p|^{-\frac{1}{2}}|d\tau|^{\frac{1}{2}}\,.$$

$\overline{T}^{\sharp}_{m,p}$ being the *primitive* period of the period trajectory of f through p (i.e., if $p_i = f^i(p)$ $i = 1, \ldots, m$ and $p, p_1, \ldots, p_{k-1}$ are all distinct but $p = p_k$ then $\overline{T}^{\sharp}_{m,p} = \overline{T}^{\sharp}_{k,p}$) . Thus these expressions are just the symbols of the oscillatory integrals

$$\hbar^{-1} a_{m,p} e^{I\imath \overline{T}^{\sharp}_{p,m} t/\hbar}$$

with $a_{m,p} = \overline{T}^{\sharp}_{m,p} |I - df^m_p|^{\frac{1}{2}}$.

Chapter 9

Differential calculus of forms, Weil's identity and the Moser trick.

The purpose of this chapter is to give a rapid review of the basics of the calculus of differential forms on manifolds. We will give two proofs of Weil's formula for the Lie derivative of a differential form: the first of an algebraic nature and then a more general geometric formulation with a "functorial" proof that we learned from Bott. We then apply this formula to the "Moser trick" and give several applications of this method.

9.1 Superalgebras.

A (commutative associative) **superalgebra** is a vector space

$$A = A_{even} \oplus A_{odd}$$

with a given direct sum decomposition into even and odd pieces, and a map

$$A \times A \to A$$

which is bilinear, satisfies the associative law for multiplication, and

$$\begin{aligned} A_{even} \times A_{even} &\to A_{even} \\ A_{even} \times A_{odd} &\to A_{odd} \\ A_{odd} \times A_{even} &\to A_{odd} \\ A_{odd} \times A_{odd} &\to A_{even} \\ \omega \cdot \sigma &= \sigma \cdot \omega \text{ if either } \omega \text{ or } \sigma \text{ are even,} \\ \omega \cdot \sigma &= -\sigma \cdot \omega \text{ if both } \omega \text{ and } \sigma \text{ are odd.} \end{aligned}$$

We write these last two conditions as

$$\omega \cdot \sigma = (-1)^{\deg \sigma \deg \omega} \sigma \cdot \omega.$$

Here deg $\tau = 0$ if τ is even, and deg $\tau = 1 \pmod 2$ if τ is odd.

9.2 Differential forms.

A **linear differential form** on a manifold, M, is a rule which assigns to each $p \in M$ a linear function on TM_p. So a linear differential form, ω, assigns to each p an element of TM_p^*. We will, as usual, only consider linear differential forms which are smooth.

The superalgebra $\Omega(M)$ is the superalgebra generated by smooth functions on M (taken as even) and by the linear differential forms, taken as odd.

Multiplication of differential forms is usually denoted by $\wedge$. The number of differential factors is called the *degree* of the form. So functions have degree zero, linear differential forms have degree one.

In terms of local coordinates, the most general *linear* differential form has an expression as $a_1 dx_1 + \cdots + a_n dx_n$ (where the a_i are functions). Expressions of the form

$$a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + \cdots + a_{n-1,n} dx_{n-1} \wedge dx_n$$

have degree two (and are even). Notice that the multiplication rules require

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

and, in particular, $dx_i \wedge dx_i = 0$. So the most general sum of products of two linear differential forms is a differential form of degree two, and can be brought to the above form, locally, after collections of coefficients. Similarly, the most general differential form of degree $k \leq n$ on an n dimensional manifold is a sum, locally, with function coefficients, of expressions of the form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad i_1 < \cdots < i_k.$$

There are $\left(\begin{array}{c} n \\ k \end{array} \right)$ such expressions, and they are all even, if k is even, and odd if k is odd.

9.3 The d operator.

There is a linear operator d acting on differential forms called *exterior* differentiation, which is completely determined by the following rules: It satisfies Leibniz' rule in the "super" form

$$d(\omega \cdot \sigma) = (d\omega) \cdot \sigma + (-1)^{\deg \omega} \omega \cdot (d\sigma).$$

On functions it is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

and, finally,

$$d(dx_i) = 0.$$

Since functions and the dx_i generate, this determines d completely. For example, on linear differential forms

$$\omega = a_1 dx_1 + \cdots a_n dx_n$$

we have

$$\begin{aligned} d\omega &= da_1 \wedge dx_1 + \cdots + da_n \wedge dx_n \\ &= \left(\frac{\partial a_1}{\partial x_1} dx_1 + \cdots \frac{\partial a_1}{\partial x_n} dx_n \right) \wedge dx_1 + \cdots \\ &\quad \left(\frac{\partial a_n}{\partial x_1} dx_1 + \cdots + \frac{\partial a_n}{\partial x_n} dx_n \right) \wedge dx_n \\ &= \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dx_1 \wedge dx_2 + \cdots + \left(\frac{\partial a_n}{\partial x_{n-1}} - \frac{\partial a_{n-1}}{\partial x_n} \right) dx_{n-1} \wedge dx_n. \end{aligned}$$

In particular, equality of mixed derivatives shows that $d^2 f = 0$, and hence that $d^2 \omega = 0$ for any differential form. Hence the rules to remember about d are:

$$\begin{aligned} d(\omega \cdot \sigma) &= (d\omega) \cdot \sigma + (-1)^{\deg \omega} \omega \cdot (d\sigma) \\ d^2 &= 0 \\ df &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \end{aligned}$$

9.4 Derivations.

A linear operator $\ell : A \to A$ is called an *odd derivation* if, like d, it satisfies

$$\ell : A_{even} \to A_{odd}, \quad \ell : A_{odd} \to A_{even}$$

and

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + (-1)^{\deg \omega} \omega \cdot \ell\sigma.$$

A linear map $\ell : A \to A$,

$$\ell : A_{even} \to A_{even}, \quad \ell : A_{odd} \to A_{odd}$$

satisfying

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + \omega \cdot (\ell\sigma)$$

is called an *even derivation.* So the Leibniz rule for derivations, even or odd, is

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + (-1)^{\deg \ell \deg \omega} \omega \cdot \ell\sigma.$$

Knowing the action of a derivation on a set of generators of a superalgebra determines it completely. For example, the equations

$$d(x_i) = dx_i, \quad d(dx_i) = 0 \quad \forall i$$

implies that

$$dp = \frac{\partial p}{\partial x_1} dx_1 + \cdots + \frac{\partial p}{\partial x_n} dx_n$$

for any polynomial, and hence determines the value of d on any differential form with polynomial coefficients. The local formula we gave for df where f is any differentiable function, was just the natural extension (by continuity, if you like) of the above formula for polynomials.

The sum of two even derivations is an even derivation, and the sum of two odd derivations is an odd derivation.

The composition of two derivations will not, in general, be a derivation, but an instructive computation from the definitions shows that the *commutator*

$$[\ell_1, \ell_2] := \ell_1 \circ \ell_2 - (-1)^{\deg \ell_1 \deg \ell_2} \ \ell_2 \circ \ell_1$$

is again a derivation which is even if both are even or both are odd, and odd if one is even and the other odd.

A derivation followed by a multiplication is again a derivation: specifically, let ℓ be a derivation (even or odd) and let τ be an even or odd element of A. Consider the map

$$\omega \mapsto \tau\ell\omega.$$

We have

$$\begin{aligned} \tau\ell(\omega\sigma) &= (\tau\ell\omega)\cdot\sigma + (-1)^{\deg\ell\deg\omega}\tau\omega\cdot\ell\sigma \\ &= (\tau\ell\omega)\cdot\sigma + (-1)^{(\deg\ell+\deg\tau)\deg\omega}\omega\cdot(\tau\ell\sigma) \end{aligned}$$

so $\omega \mapsto \tau\ell\omega$ is a derivation whose degree is

$$\deg\tau + \deg\ell.$$

9.5 Pullback.

Let $\phi : M \to N$ be a smooth map. Then the pullback map ϕ^* is a linear map that sends differential forms on N to differential forms on M and satisfies

$$\begin{aligned} \phi^*(\omega \wedge \sigma) &= \phi^*\omega \wedge \phi^*\sigma \\ \phi^* d\omega &= d\phi^*\omega \\ (\phi^* f) &= f \circ \phi. \end{aligned}$$

The first two equations imply that ϕ^* is completely determined by what it does on functions. The last equation says that on functions, ϕ^* is given by "substitution": In terms of local coordinates on M and on N ϕ is given by

$$\begin{aligned} \phi(x^1, \ldots, x^m) &= (y^1, \ldots, y^n) \\ y^i &= \phi^i(x^1, \ldots, x^m) \quad i = 1, \ldots, n \end{aligned}$$

where the ϕ_i are smooth functions. The local expression for the pullback of a function $f(y^1,\ldots,y^n)$ is to substitute ϕ^i for the y^is as into the expression for f so as to obtain a function of the x's.

It is important to observe that the pull back on differential forms is defined for any smooth map, not merely for diffeomorphisms. This is the great advantage of the calculus of differential forms.

9.6 Chain rule.

Suppose that $\psi : N \to P$ is a smooth map so that the composition

$$\phi \circ \psi : M \to P$$

is again smooth. Then the *chain rule* says

$$(\phi \circ \psi)^* = \psi^* \circ \phi^*.$$

On functions this is essentially a tautology - it is the associativity of composition: $f \circ (\phi \circ \psi) = (f \circ \phi) \circ \psi$. But since pull-back is completely determined by what it does on functions, the chain rule applies to differential forms of any degree.

9.7 Lie derivative.

Let ϕ_t be a one parameter group of transformations of M. If ω is a differential form, we get a family of differential forms, $\phi_t^*\omega$ depending differentiably on t, and so we can take the derivative at $t = 0$:

$$\frac{d}{dt}\left(\phi_t^*\omega\right)_{|t=0} = \lim_{t=0} \frac{1}{t}\left[\phi_t^*\omega - \omega\right].$$

Since $\phi_t^*(\omega\wedge\sigma) = \phi_t^*\omega\wedge\phi_t^*\sigma$ it follows from the Leibniz argument that

$$\ell_\phi :\ \omega \mapsto \frac{d}{dt}\left(\phi_t^*\omega\right)_{|t=0}$$

is an even derivation. We want a formula for this derivation.

Notice that since $\phi_t^* d = d\phi_t^*$ for all t, it follows by differentiation that

$$\ell_\phi d = d\ell_\phi$$

and hence the formula for ℓ_ϕ is completely determined by how it acts on functions.

Let X be the vector field generating ϕ_t. Recall that the geometrical significance of this vector field is as follows: If we fix a point x, then

$$t \mapsto \phi_t(x)$$

is a curve which passes through the point x at $t = 0$. The tangent to this curve at $t = 0$ is the vector $X(x)$. In terms of local coordinates, X has coordinates $X = (X^1, \ldots, X^n)$ where $X^i(x)$ is the derivative of $\phi^i(t, x^1, \ldots, x^n)$ with respect to t at $t = 0$. The chain rule then gives, for any function f,

$$\begin{aligned} \ell_\phi f &= \frac{d}{dt} f(\phi^1(t, x^1, \ldots, x^n), \ldots, \phi_n(t, x^1, \ldots, x^n))_{|t=0} \\ &= X^1 \frac{\partial f}{\partial x_1} + \cdots + X^n \frac{\partial f}{\partial x_n}. \end{aligned}$$

For this reason we use the notation

$$X = X^1 \frac{\partial}{\partial x_1} + \cdots + X^n \frac{\partial}{\partial x_n}$$

so that the differential operator

$$f \mapsto Xf$$

gives the action of ℓ_ϕ on functions.

As we mentioned, this action of ℓ_ϕ on functions determines it completely. In particular, ℓ_ϕ depends only on the vector field X, so we may write

$$\ell_\phi = D_X$$

where D_X is the even derivation determined by

$$D_X f = Xf, \quad D_X d = dD_X.$$

9.8 Weil's formula.

But we want a more explicit formula for D_X. For this it is useful to introduce an odd derivation associated to X called the *interior product* and denoted by $i(X)$. It is defined as follows: First consider the case where

$$X = \frac{\partial}{\partial x_j}$$

and define its interior product by

$$i\left(\frac{\partial}{\partial x_j}\right) f = 0$$

for all functions while

$$i\left(\frac{\partial}{\partial x_j}\right) dx_k = 0, \quad k \neq j$$

and

$$i\left(\frac{\partial}{\partial x_j}\right) dx_j = 1.$$

The fact that it is a derivation then gives an easy rule for calculating $i(\partial/\partial x_j)$ when applied to any differential form: Write the differential form as

$$\omega + dx_j \wedge \sigma$$

where the expressions for ω and σ do not involve dx_j. Then

$$i\left(\frac{\partial}{\partial x_j}\right)[\omega + dx_j \wedge \sigma] = \sigma.$$

The operator

$$X^j i\left(\frac{\partial}{\partial x_j}\right)$$

which means first apply $i(\partial/\partial x_j)$ and then multiply by the function X^j is again an odd derivation, and so we can make the definition

$$i(X) := X^1 i\left(\frac{\partial}{\partial x_1}\right) + \cdots + X^n i\left(\frac{\partial}{\partial x_n}\right). \quad (9.1)$$

It is easy to check that this does not depend on the local coordinate system used.

Notice that we can write

$$Xf = i(X)df.$$

In particular we have

$$\begin{aligned} D_X dx_j &= dD_X x_j \\ &= dX_j \\ &= di(X)dx_j. \end{aligned}$$

We can combine these two formulas as follows: Since $i(X)f = 0$ for any function f we have

$$D_X f = di(X)f + i(X)df.$$

Since $ddx_j = 0$ we have

$$D_X dx_j = di(X)dx_j + i(X)ddx_j.$$

Hence

$$D_X = di(X) + i(X)d = [d, i(X)] \tag{9.2}$$

when applied to functions or to the forms dx_j. But the right hand side of the preceding equation is an even derivation, being the commutator of two odd derivations. So if the left and right hand side agree on functions and on the differential forms dx_j they agree everywhere. This equation, (9.2), known as *Weil's formula*, is a basic formula in differential calculus.

We can use the interior product to consider differential forms of degree k as $k-$multilinear functions on the tangent space at each point. To illustrate, let σ be a differential form of degree two. Then for any vector field, X, $i(X)\sigma$ is a linear differential form, and hence can be evaluated on any vector field, Y to produce a function. So we define

$$\sigma(X,Y) := [i(X)\sigma](Y).$$

We can use this to express exterior derivative in terms of ordinary derivative and Lie bracket: If θ is a linear differential form, we have

$$\begin{aligned} d\theta(X,Y) &= [i(X)d\theta](Y) \\ i(X)d\theta &= D_X\theta - d(i(X)\theta) \\ d(i(X)\theta)(Y) &= Y[\theta(X)] \\ [D_X\theta](Y) &= D_X[\theta(Y)] - \theta(D_X(Y)) \\ &= X[\theta(Y)] - \theta([X,Y]) \end{aligned}$$

where we have introduced the notation $D_XY =: [X,Y]$ which is legitimate since on functions we have

$$(D_XY)f = D_X(Yf) - YD_Xf = X(Yf) - Y(Xf)$$

so D_XY as an operator on functions is exactly the commutator of X and Y. (See below for a more detailed geometrical interpretation of D_XY.) Putting the previous pieces together gives

$$d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y]), \tag{9.3}$$

with similar expressions for differential forms of higher degree.

9.9 Integration.

Let

$$\omega = f dx_1 \wedge \cdots \wedge dx_n$$

be a form of degree n on $\mathbf{R}^n$. (Recall that the most general differential form of degree n is an expression of this type.) Then its integral is defined by

$$\int_M \omega := \int_M f dx_1 \cdots dx_n$$

where M is any (measurable) subset. This, of course is subject to the condition that the right hand side converges if M is unbounded. There is a lot of hidden subtlety built into this definition having to do with the notion of orientation. But for the moment this is a good working definition.

The *change of variables formula* says that if $\phi : M \to \mathbf{R}^n$ is a smooth differentiable map which is one to one whose Jacobian determinant is everywhere positive, then

$$\int_M \phi^* \omega = \int_{\phi(M)} \omega.$$

9.10 Stokes theorem.

Let U be a region in $\mathbf{R}^n$ with a chosen orientation and smooth boundary. We then orient the boundary according to the rule that an outward pointing normal vector, together with the a positive frame on the boundary give a positive frame in $\mathbf{R}^n$. If σ is an $(n-1)-$form, then

$$\int_{\partial U} \sigma = \int_U d\sigma.$$

A manifold is called *orientable* if we can choose an atlas consisting of charts such that the Jacobian of the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ is always positive. Such a choice of an atlas is called an orientation. (Not all manifolds are orientable.) If we have chosen an orientation, then relative to the charts of our orientation, the transition laws for an $n-$form (where n = dim M) and for a density are the same. In other words, given an orientation, we can identify densities with $n-$forms and $n-$form with densities. Thus we may integrate $n-$forms. The change of variables formula then holds for orientation preserving diffeomorphisms as does Stokes theorem.

9.11 Lie derivatives of vector fields.

Let Y be a vector field and ϕ_t a one parameter group of transformations whose "infinitesimal generator" is some other vector field X. We can consider the "pulled back" vector field ϕ_t^*Y defined by

$$\phi_t^*Y(x) = d\phi_{-t}\{Y(\phi_t x)\}.$$

In words, we evaluate the vector field Y at the point $\phi_t(x)$, obtaining a tangent vector at $\phi_t(x)$, and then apply the differential of the (inverse) map ϕ_{-t} to obtain a tangent vector at x.

If we differentiate the one parameter family of vector fields ϕ_t^*Y with respect to t and set $t = 0$ we get a vector field which we denote by $D_X Y$:

$$D_X Y := \frac{d}{dt}\phi_t^* Y_{|t=0}.$$

If ω is a linear differential form, then we may compute $i(Y)\omega$ which is a function whose value at any point is obtained by evaluating the linear function $\omega(x)$ on the tangent vector $Y(x)$. Thus

$$i(\phi_t^*Y)\phi_t^*\omega(x) = \langle (d(\phi_t)_x)^*\omega(\phi_t x), d\phi_{-t}Y(\phi_t x)\rangle = \{i(Y)\omega\}(\phi_t x).$$

In other words,

$$\phi_t^*\{i(Y)\omega\} = i(\phi_t^*Y)\phi_t^*\omega.$$

We have verified this when ω is a differential form of degree one. It is trivially true when ω is a differential form of degree zero, i.e. a function, since then both sides are zero. But then, by the derivation property, we conclude that it is true for forms of all degrees. We may rewrite the result in shorthand form as

$$\phi_t^* \circ i(Y) = i(\phi_t^*Y) \circ \phi_t^*.$$

Since $\phi_t^* d = d\phi_t^*$ we conclude from Weil's formula that

$$\phi_t^* \circ D_Y = D_{\phi_t^*Y} \circ \phi_t^*.$$

Until now the subscript t was superfluous, the formulas being true for any fixed diffeomorphism. Now we differentiate the preceding equations with respect to t and set $t = 0$. We obtain,using Leibniz's rule,

$$D_X \circ i(Y) = i(D_X Y) + i(Y) \circ D_X$$

and

$$D_X \circ D_Y = D_{D_X Y} + D_Y \circ D_X.$$

This last equation says that Lie derivative (on forms) with respect to the vector field $D_X Y$ is just the commutator of D_X with D_Y:

$$D_{D_X Y} = [D_X, D_Y].$$

For this reason we write

$$[X, Y] := D_X Y$$

and call it the Lie bracket (or commutator) of the two vector fields X and Y. The equation for interior product can then be written as

$$i([X, Y]) = [D_X, i(Y)].$$

The Lie bracket is antisymmetric in X and Y. We may multiply Y by a function g to obtain a new vector field gY. Form the definitions we have

$$\phi_t^*(gY) = (\phi_t^* g)\phi_t^* Y.$$

Differentiating at $t = 0$ and using Leibniz's rule we get

$$[X, gY] = (Xg)Y + g[X, Y] \tag{9.4}$$

where we use the alternative notation Xg for $D_X g$. The antisymmetry then implies that for any differentiable function f we have

$$[fX, Y] = -(Yf)X + f[X, Y]. \tag{9.5}$$

From both this equation and from Weil's formula (applied to differential forms of degree greater than zero) we see that the Lie derivative with respect to X at a point x depends on more than the value of the vector field X at x.

9.12 Jacobi's identity.

From the fact that $[X, Y]$ acts as the commutator of X and Y it follows that for any three vector fields X, Y and Z we have

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

This is known as **Jacobi's identity**. We can also derive it from the fact that $[Y, Z]$ is a natural operation and hence for any one parameter group ϕ_t of diffeomorphisms we have

$$\phi_t^*([Y, Z]) = [\phi_t^* Y, \phi_t^* Z].$$

If X is the infinitesimal generator of ϕ_t then differentiating the preceding equation with respect to t at $t = 0$ gives

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

In other words, X acts as a derivation of the "mutliplication" given by Lie bracket. This is just Jacobi's identity when we use the antisymmetry of the bracket. In the future we we will have occasion to take cyclic sums such as those which arise on the left of Jacobi's identity. So if F is a function of three vector fields (or of three elements of any set) with values in some vector space (for example in the space of vector fields) we will define the cyclic sum $\mathcal{C}yc\ F$ by

$$\mathcal{C}yc\, F(X, Y, Z) := F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y).$$

With this definition Jacobi's identity becomes

$$\mathcal{C}yc\ [X, [Y, Z]] = 0. \tag{9.6}$$

9.13 A general version of Weil's formula.

Let W and Z be differentiable manifolds, let I denote an interval on the real line containing the origin, and let

$$\phi : W \times I \to Z$$

be a smooth map. We let $\phi_t : W \to Z$ be defined by

$$\phi_t(w) := \phi(w, t).$$

We think of ϕ_t as a one parameter family of maps from W to Z. We let ξ_t denote the tangent vector field along ϕ_t. In more detail:

$$\xi_t : W \to TZ$$

is defined by letting $\xi_t(w)$ be the tangent vector to the curve $s \mapsto \phi(w, s)$ at $s = t$.

If σ is a differential form on Z of degree $k + 1$, we let the expression $\phi_t^* i(\xi_t)\sigma$ denote the differential form on W of degree k whose value at tangent vectors $\eta_1, \ldots, \eta_k$ at $w \in W$ is given by

$$\phi_t^* i(\xi_t)\sigma(\eta_1, \ldots, \eta_k) := (i(\xi_t)(w))\sigma)(d(\phi_t)_w \eta_1, \ldots, d(\phi_t)_w \eta_k). \tag{9.7}$$

It is only the combined expression $\phi_t^* i(\xi_t)\sigma$ which will have any sense in general: since ξ_t is not a vector field on Z, the expression $i(\xi_t)\sigma$ will not make sense as a stand alone object (in general).

Let σ_t be a smooth one-parameter family of differential forms on Z. Then

$$\phi_t^* \sigma_t$$

is a smooth one parameter family of forms on W, which we can then differentiate with respect to t. The general form of Weil's formula is:

$$\frac{d}{dt}\phi_t^* \sigma_t = \phi_t^* \frac{d\sigma_t}{dt} + \phi_t^* i(\xi_t) d\sigma + d\phi_t^* i(\xi_t)\sigma. \tag{9.8}$$

Before proving the formula, let us note that it is functorial in the following sense: Suppose that that $F : X \to W$ and $G : Z \to Y$ are smooth maps, and that τ_t is a smooth family of differential forms on Y. Suppose that $\sigma_t = G^* \tau_t$ for all t. We can consider the maps

$$\psi_t : X \to Y, \quad \psi_t := G \circ \phi_t \circ F$$

and then the smooth one parameter familiy of differential forms

$$\psi_t^* \tau_t$$

on X. The tangent vector field ζ_t along ψ_t is given by

$$\zeta_t(x) = dG_{\phi_t(F(x))}\left(\xi_t(F(x))\right).$$

So

$$\psi_t^* i(\zeta_t)\tau_t = F^*\left(\phi_t^* i(\xi_t) G^* \tau_t\right).$$

Therefore, if we know that (9.8) is true for ϕ_t and σ_t, we can conclude that the analogous formula is true for ψ_t and τ_t.

Consider the special case of (9.8) where we take the one parameter family of maps

$$f_t : W \times I \to W \times I, \qquad f_t(w,s) = (w, s+t).$$

Let

$$G : W \times I \to Z$$

be the map ϕ, and let

$$F : W \to W \times I$$

be the map

$$F(w) = (w, 0).$$

Then

$$(G \circ f_t \circ F)(w) = \phi_t(w).$$

Thus the functoriality of the formula (9.8) shows that we only have to prove it for the special case $\phi_t = f_t : W \times I \to W \times I$ as given above!

In this case, it is clear that the vector field ξ_t along ψ_t is just the constant vector field $\frac{\partial}{\partial s}$ evaluated at $(x, s+t)$. The most general differential (t-dependent) on $W \times I$ can be written as

$$ds \wedge a + b$$

where a and b are differential forms on W. (In terms of local coordinates $s, x^1, \ldots, x^n$ these forms a and b are sums of terms that have the expression

$$cdx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where c is a function of s, t and x.) To show the full dependence on the variables we will write

$$\sigma_t = ds \wedge a(x,s,t)dx + b(x,s,t)dx.$$

With this notation it is clear that

$$\phi_t^*\sigma_t = ds \wedge a(x, s+t, t)dx + b(x, s+t, t)dx$$

and therefore

$$\frac{d\phi_t^*\sigma_t}{dt} = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx + \frac{\partial b}{\partial s}(x, s+t, t)dx$$

$$+ds\wedge\frac{\partial a}{\partial t}(x,s+t,t)dx+\frac{\partial b}{\partial t}(x,s+t,t)dx.$$

So

$$\frac{d\phi_t^*\sigma_t}{dt}-\phi_t^*\frac{d\sigma_t}{dt}=ds\wedge\frac{\partial a}{\partial s}(x,s+t,t)dx+\frac{\partial b}{\partial s}(x,s+t,t)dx.$$

Now

$$i\left(\frac{\partial}{\partial s}\right)\sigma_t=adx$$

so

$$\phi_t^*i(\xi_t)\sigma_t=a(x,s+t,t)dx.$$

Therefore

$$d\phi_t^*i(\xi_t)\sigma_t=ds\wedge\frac{\partial a}{\partial s}(x,s+t,t)dx+d_W(a(x,s+t,t)dx).$$

Also

$$d\sigma_t=-ds\wedge d_W(adx)+\frac{\partial b}{\partial s}ds\wedge dx+d_Wbdx$$

so

$$i\left(\frac{\partial}{\partial s}\right)d\sigma_t=-d_W(adx)+\frac{\partial b}{\partial s}dx$$

and therefore

$$\phi_t^*i(\xi_t)d\sigma_t=-d_Wa(x,s+t,t)dx+\frac{\partial b}{\partial s}(x,s+t,t)dx.$$

So

$$d\phi_t^*i(\xi_t)\sigma_t+\phi_t^*i(\xi_t)d\sigma_t=ds\wedge\frac{\partial a}{\partial s}(x,s+t,t)dx+\frac{\partial b}{\partial s}(x,s+t,t)dx$$

$$=\frac{d\phi_t^*\sigma_t}{dt}-\phi_t^*\frac{d\sigma_t}{dt}$$

proving (9.8).

A special case of (9.8) is the following. Suppose that $W=Z=M$ and ϕ_t is a family of diffeomorphisms $f_t:M\to M$. Then ξ_t is given by

$$\xi_t(p)=v_t(f_t(p))$$

where v_t is the vector field

$$v_t(f(p))=\frac{d}{dt}f_t(p).$$

In this case $i(v_t)\sigma_t$ makes sense, and so we can write (9.8) as

$$\frac{d\phi_t^*\sigma_t}{dt}=\phi_t^*\frac{d\sigma_t}{dt}+\phi_t^*D_{v_t}\sigma_t. \tag{9.9}$$

9.14 The Moser trick.

Let M be a differentiable manifold and let ω_0 and ω_1 be smooth k-forms on M. Let us examine the following question: does there exist a diffeomorphism $f : M \to M$ such that $f^*\omega_1 = \omega_0$?

Moser answers this kind of question by making it harder! Let $\omega_t, \quad 0 \le t \le 1$ be a family of k-forms with $\omega_t = \omega_0$ at $t = 0$ and $\omega_t = \omega_1$ at $t = 1$. We look for a one parameter family of diffeomorphisms

$$f_t : M \to M, \quad 0 \le t \le 1$$

such that

$$f_t^*\omega_t = \omega_0 \tag{9.10}$$

and

$$f_0 = \mathrm{id}\,.$$

Let us differentiate (9.10) with respect to t and apply (9.9). We obtain

$$f_t^*\dot{\omega}_t + f_t^* D_{v_t}\omega_t = 0$$

where we have written $\dot{\omega}_t$ for $\frac{d\omega_t}{dt}$. Since f_t is required to be a diffeomorphism, this becomes the requirement that

$$D_{v_t}\omega_t = -\dot{\omega}_t. \tag{9.11}$$

Moser's method is to use "geometry" to solve this equation for v_t if possible. Once we have found v_t, solve the equations

$$\frac{d}{dt} f_t(p) = v_t(f_t(p), \qquad f_0(p) = p \tag{9.12}$$

for f_t. Notice that for p fixed and $\gamma(t) = f_t(p)$ this is a system of ordinary differential equations

$$\frac{d}{dt}\gamma(t) = v_t(\gamma(t)), \quad \gamma(0) = p.$$

The standard existence theorems for ordinary differential equations guarantees the existence of of a solution depending smoothly on p at least for $|t| < \epsilon$. One then must make some additional hypotheses that guarantee existence for all time (or at least up to $t = 1$). Two such additional hypotheses might be

- M is compact, or

- C is a closed subset of M on which $v_t \equiv 0$. Then for $p \in C$ the solution for all time is $f_t(p) = p$. Hence for p close to C solutions will exist for a long time. Under this condition there will exist a neighborhood U of C and a family of diffeomorphisms

  $$f_t : U \to M$$

  defined for $0 \le t \le 1$ such

  $$f_0 = \mathrm{id}, \qquad f_{t|C} = \mathrm{id}\,\forall t$$

  and (9.10) is satisfied.

We now give some illustrations of the Moser trick.

9.14.1 Volume forms.

Let M be a compact oriented connected n-dimensional manifold. Let ω_0 and ω_1 be nowhere vanishing n-forms with the same volume:

$$\int_M \omega_0 = \int_M \omega_1.$$

Moser's theorem asserts that under these conditions there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Moser invented his method for the proof of this theorem.

The first step is to choose the ω_t. Let

$$\omega_t := (1-t)\omega_0 + t\omega_1.$$

Since both ω_0 and ω_1 are nowhere vanishing, and since they yield the same integral (and since M is connected), we know that at every point they are either both positive or both negative relative to the orientation. So ω_t is nowhere vanishing. Clearly $\omega_t = \omega_0$ at $t = 0$ and $\omega_t = \omega_1$ at $t = 1$. Since $d\omega_t = 0$ as ω_t is an n-from on an n-dimensional manifold,

$$D_{v_t}\omega_t = di(v_t)\omega_t$$

by Weil's formula. Also

$$\dot{\omega}_t = \omega_1 - \omega_0.$$

Since $\int_M \omega_0 = \int_M \omega_1$ we know that

$$\omega_0 - \omega_1 = d\nu$$

for some $(n-1)$-form ν. Thus (9.11) becomes

$$di(v_t)\omega_t = d\nu.$$

We will certainly have solved this equation if we solve the harder equation

$$i(v_t)\omega_t = \nu.$$

But this equation has a unique solution since ω_t is no-where vanishing. QED

9.14.2 Variants of the Darboux theorem.

We present these in Chapter 2.

9.14.3 The classical Morse lemma.

Let $M = \mathbb{R}^n$ and $\phi_i \in C^\infty(\mathbb{R}^n)$, $i = 0, 1$. Suppose that 0 is a non-degenerate critical point for both ϕ_0 and ϕ_1, suppose that $\phi_0(0) = \phi_1(0) = 0$ and that they have the same Hessian at 0, i.e. suppose that

$$\left(d^2\phi_0\right)(0) = \left(d^2\phi_1\right)(0).$$

The Morse lemma asserts that there exist neighborhoods U_0 and U_1 of 0 in $\mathbb{R}^n$ and a diffeomorphism

$$f : U_0 \to U_1, \quad f(0) = 0$$

such that

$$f^*\phi_1 = \phi_0.$$

Proof. Set

$$\phi_t := (1-t)\phi_0 + t\phi_1.$$

The Moser trick tells us to look for a vector field v_t with

$$v_t(0) = 0, \quad \forall\ t$$

and

$$D_{v_t}\phi_t = -\dot{\phi}_t = \phi_0 - \phi_1.$$

The function ϕ_t has a non-degenerate critical point at zero with the same Hessian as ϕ_0 and ϕ_1 and vanishes at 0. Thus for each fixed t, the functions

$$\frac{\partial \phi_t}{\partial x^i}$$

form a system of coordinates about the origin.

If we expand v_t in terms of the standard coordinates

$$v_t = \sum_j v_j(x,t) \frac{\partial}{\partial x^j}$$

then the condition $v_j(0,t) = 0$ implies that we must be able to write

$$v_j(x,t) = \sum_i v_{ij}(x,t) \frac{\partial \phi_t}{\partial x^i}.$$

for some smooth functions v_{ij}. Thus

$$D_{v_t}\phi_t = \sum_{ij} v_{ij}(x,t) \frac{\partial \phi_t}{\partial x^i} \frac{\partial \phi_t}{\partial x^j}.$$

Similarly, since $-\dot{\phi}_t$ vanishes at the origin together with its first derivatives, we can write

$$-\dot{\phi}_t = \sum_{ij} h_{ij} \frac{\partial \phi_t}{\partial x^i} \frac{\partial \phi_t}{\partial x^j}$$

where the h_{ij} are smooth functions. So the Moser equation $D_{v_t}\phi_t = -\dot{\phi}_t$ is satisfied if we set

$$v_{ij}(x,t) = h_{ij}(x,t).$$

Notice that our method of proof shows that if the ϕ_i depend smoothly on some paramters lying in a compact manifold S then the diffeomorphism f can be chosen so as to depend smoothly on $s \in S$.

In Section 5.11 we give a more refined version of this argument to prove the Hörmander-Morse lemma for generating functions.

In differential topology books the classical Morse lemma is usually stated as follows:

Theorem 43 *Let M be a manifold and $\phi : M \to \mathbb{R}$ be a smooth function. Suppose that $p \in M$ is a non-degenerate critical point of ϕ and that the signature* *of* $d^2\phi_p$ *is* $(k, n-k)$. *Then there exists a system of coordinates* $(U, x_1, \ldots, x_n)$ *centered at* p *such that in this coordinate system*

$$\phi = c + \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2.$$

Proof. Choose any coordinate system $(W, y_1, \ldots y_n)$ centered about p and apply the previous result to

$$\phi_1 = \phi - c$$

and

$$\phi_0 = \sum h_{ij} y_i y_j$$

where

$$h_{ij} = \frac{\partial^2 \phi}{\partial y_i \partial y_i}(0).$$

This gives a change of coordinates in terms of which $\phi - c$ has become a non-degenerate quadratic form. Now apply Sylvester's theorem in linear algebra which says that a linear change of variables can bring such a non-degenerate quadratic form to the desired diagonal form.

Chapter 10

The method of stationary phase

10.1 Gaussian integrals.

We recall a basic computation in the integral calculus:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1. \tag{10.1}$$

This is proved by taking the square of the left hand side and then passing to polar coordinates:

$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx\right]^2 =$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}dxdy \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}rdrd\theta \\
&= \int_0^{\infty} e^{-r^2/2}rdr \\
&= 1.
\end{aligned}$$

10.1.1 The Fourier transform of a Gaussian.

Now

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-\eta x}dx$$

converges for all complex values of η, uniformly in any compact region. Hence it defines an analytic function which may be evaluated by taking η to be real and then using analytic continuation. For real η we complete the square and make a change of variables:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-\frac{x^2}{2}-x\eta)dx =$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\frac{1}{2}(-(x+\eta)^2+\eta^2)dx \\
&= \exp(\eta^2/2)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-(x^2+\eta^2)/2)dx \\
&= \exp(\eta^2/2).
\end{aligned}$$

As we mentioned, this equation is true for any complex value of η. In particular, setting $\eta = -i\xi$ we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-x^2/2+i\xi x)dx = \exp(-\xi^2/2). \tag{10.2}$$

In short,

1 *The Fourier transform of the Gaussian function* $x \mapsto \exp(-x^2/2)$ *is* $\xi \mapsto e^{-\xi^2/2}$.

If f is any smooth function vanishing rapidly at infinity, and $\hat{f}$ denotes its Fourier transform, then the Fourier transform of $x \mapsto f(cx)$ is $\xi \mapsto \frac{1}{c}\hat{f}(\xi/c)$. In particular, if we take $\lambda > 0,\ c = \lambda^{\frac{1}{2}}$ we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-\lambda x^2/2+i\xi x)dx = \left(\frac{1}{\lambda}\right)^{\frac{1}{2}}\exp(-\xi^2/2\lambda). \tag{10.3}$$

We proved this formula for λ real and positive. But the integral on the left makes sense for all λ with Re $\lambda > 0$, and hence this formula remains true in the entire open right hand plane Re $\lambda > 0$, provided we interpret the square root occurring on the right as arising by analytic continuation from the positive real axis.

We can say more: The integral on the left converges uniformly (but not absolutely) for λ in any region of the form

$$\text{Re } \lambda \geq 0, \quad |\lambda| > \delta > 0.$$

To see this, observe that for any $S > R > 0$ we have

$$e^{-\lambda x^2/2} = -\frac{1}{\lambda x}\frac{d}{dx}\exp(-\lambda x^2/2) \quad \text{for } R \le x \le S$$

so we can apply integration by parts to get

$$\int_R^S e^{-\lambda x^2/2} e^{i\xi x} dx =$$

$$\frac{1}{\lambda}\left(\frac{1}{R}e^{-\lambda R^2/2+i\xi R} - \frac{1}{S}e^{-S^2/2+i\xi S} + \int_R^S e^{-\lambda x^2/2}\frac{d}{dx}\left(\frac{e^{i\xi x}}{x}\right)dx\right)$$

and integrate by parts once more to bound the integral on the right. We conclude that

$$\left|\int_R^S e^{-\lambda x^2/2} e^{i\xi x} dx\right| = O(\frac{1}{|\lambda R|}).$$

10.2 The integral $\int e^{-\lambda x^2/2} h(x) dx$.

This same argument shows that

$$\int e^{-\lambda x^2/2} h(x) dx$$

is convergent for any h with two bounded continuous derivatives. Indeed,

$$\int_R^S e^{-\lambda x^2/2} h(x) dx =$$

$$\begin{aligned}
&= -\frac{1}{\lambda}\int_R^S \frac{h(x)}{x}\frac{d}{dx}e^{-\lambda x^2/2}dx \\
&= -\lambda^{-1}e^{-\lambda x^2/2}(h(x)/x)\Big|_R^S \\
&\quad +\frac{1}{\lambda}\int_R^S e^{-\lambda x^2/2}\frac{d}{dx}(h(x)/x)dx \\
&= -\lambda^{-2}e^{-\lambda x^2/2}[\lambda(h(x)/x) - (1/x)\frac{d}{dx}(h(x)/x)]\Big|_R^S \\
&\quad +\lambda^{-2}\int_R^S e^{-\lambda x^2/2}[(1/x)(h(x)/x)']'dx.
\end{aligned}$$

This last integral is absolutely convergent, and the boundary terms tend to zero as $R \to \infty$.

This argument shows that if M is a bound for h and its first two derivatives, the above expressions can all be estimated purely in terms of M. Thus if h depends on some auxiliary parameters, and is uniformly bounded together with its first two derivatives with respect to these parameters, then the integral $\int_{-\infty}^{\infty} h(x)\exp(-\lambda x^2/2)dx$ converges uniformly with respect to these parameters.

Let us push this argument one step further. Suppose that h has derivatives of all order which are bounded on the entire real axis, and suppose further that $h \equiv 0$ in some neighborhood, $|x| < \epsilon$, of the origin. If we do the integration by parts

$$\int_R^S e^{-\lambda x^2/2} h(x)dx$$

$$= -\lambda^{-1} e^{-\lambda x^2/2}(h(x)/x)\Big|_R^S + \frac{1}{\lambda}\int_R^S e^{-\lambda x^2/2}\frac{d}{dx}(\frac{h(x)}{x})dx,$$

choose $R < \epsilon$ and let $S \to \infty$. We conclude that

$$\int_{-\infty}^{\infty} e^{-\lambda x^2/2} h(x)dx = \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\lambda x^2/2}\frac{d}{dx}(h(x)/x)dx.$$

The right hand side is a function of the same sort as h. We conclude that

$$\int_{\mathbb{R}} e^{-\lambda x^2/2} h(x)dx = O(\lambda^{-N})$$

for all N if h vanishes in some neighborhood of the origin has derivatives of all order which are each bounded on the entire line.

10.3 Gaussian integrals in n dimensions.

Getting back to the case $h \equiv 1$, if we take $\lambda = \mp ir, \quad r > 0$ and set $\xi = 0$ in (10.3) then analytic continuation from the positive real axis gives $\lambda^{\frac{1}{2}} = e^{\mp\pi i/4}$ and we obtain

$$\int_{-\infty}^{\infty} e^{\pm irx^2/2}dx = \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{\pm\pi i/4}. \qquad (10.4)$$

Doing the same computation in n - dimensions gives

$$\int e^{i\tau Q/2}dy = \left(\frac{2\pi}{\tau}\right)^{\frac{n}{2}} \left(\frac{1}{r_1 \cdot r_2 \cdots r_n}\right)^{\frac{1}{2}} e^{i \operatorname{sgn} Q\pi/4} \tag{10.5}$$

if

$$Q(y) = \sum \pm r_i (y^i)^2.$$

Now $r_1 \cdot r_2 \cdots r_n = |\det Q|$. So we can rewrite the above equation as

$$\int e^{i\tau Q/2}dy = \left(\frac{2\pi}{\tau}\right)^{\frac{n}{2}} \frac{1}{\sqrt{|\det Q|}} e^{i \operatorname{sgn} Q\pi/4} \tag{10.6}$$

We proved this formula under the assumption that Q was in diagonal form. But if Q is any non-degenerate quadratic form, we know that there is an orthogonal change of coordinates which brings Q to diagonal form. By this change of variables we see that

2 *(10.6) is valid for any non-degenerate quadratic form.*

10.4 Using the multiplication formula for the Fourier transform.

Recall that in one dimension this says that if $f, g \in \mathcal{S}(\mathbb{R})$ and $\hat{f}, \hat{g}$ denote their Fourier transforms then

$$\int_{\mathbb{R}} \hat{f}(\xi) g(\xi) d\xi = \int_{\mathbb{R}} f(x)\hat{g}(x)dx.$$

In this formula let us take

$$g(\xi) = e^{-\frac{\xi^2}{2\lambda}}$$

where Re $\lambda > 0$ so that

$$\hat{g}(x) = \lambda^{\frac{1}{2}} e^{-\lambda x^2/2}$$

where the square root is given by the positive square root on the positive axis and extended by analytic continuation. So the multiplication formula yields

$$\int_{\mathbb{R}} \hat{f}(\xi) e^{-\frac{\xi^2}{2\lambda}} d\xi = \lambda^{\frac{1}{2}} \int_{\mathbb{R}} f(x) e^{-\frac{\lambda x^2}{2}} dx.$$

Take
$$\lambda = \epsilon - ia, \quad \epsilon > 0, \quad a \in \mathbb{R} - \{0\}$$
and let $\epsilon \searrow 0$. We get
$$\int_{\mathbb{R}} \hat{f}(\xi) e^{-\frac{i\xi^2}{2a}} = |a|^{\frac{1}{2}} e^{-\frac{\pi i}{4} \operatorname{sgn} a} \int_{\mathbb{R}} f(x) e^{\frac{iax^2}{2}} dx$$
which we can rewrite as
$$\int_{\mathbb{R}} f(x) e^{i\frac{ax^2}{2}} dx = |a|^{-\frac{1}{2}} e^{\frac{\pi i}{4}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-\frac{i\xi^2}{2a}} d\xi.$$

We can pass from this one dimensional formula to an n - dimensional formula as follows: Let $A = (a_{k\ell})$ be a non-singular symmetric $n \times n$ matrix and let $\operatorname{sgn} A$ denote the signature of the quadatic form
$$Q(x) = \langle Ax, x \rangle = \sum a_{ij} x_i x_j.$$
Let
$$B := A^{-1}.$$
Then for any $t > 0$ we have
$$\int_{\mathbb{R}^n} f(x) e^{i\frac{t}{2}\langle Ax,x\rangle} dx = t^{-\frac{n}{2}} |\det A|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sgn} A} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\frac{i}{2t}\langle B\xi,\xi\rangle} d\xi. \tag{10.7}$$
The proof is via diagonalization as before. We may make an orthogonal change of coordinates relative to which A becomes diagonal. Then if f is a product function
$$f(x_1, \ldots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n)$$
the formula reduces to the one dimensional formula we have already proved. Since the linear combination of these functions are dense, the formula is true in general.

10.5 A local version of stationary phase.

In order to conform with standard notation let us set $t = \hbar^{-1}$ in (10.7). The right hand side of (10.7) becomes
$$\hbar^{\frac{n}{2}} |\det A|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sgn} A} a(\hbar)$$

where

$$a(\hbar) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-i\frac{\hbar}{2}\langle B\xi,\xi\rangle} d\xi.$$

Let us now use the Taylor formula for the exponential:

$$\left| e^{ix} - \sum_{k=0}^{m} \frac{(ix)^k}{k!} \right| \leq \frac{|x|^{m+1}}{(m+1)!}.$$

Thus the function a can be estimated by the sum

$$\sum_{k=0}^{m} \frac{1}{k!} \left(-\frac{i\hbar}{2} \right)^k \int_{\mathbb{R}^n} \langle B\xi,\xi\rangle^k \hat{f}(\xi) d\xi$$

with an error that is bounded by

$$\frac{1}{(m+1)!} \left(\frac{\hbar}{2} \right)^{m+1} \int_{\mathbb{R}^n} \left| \langle B\xi,\xi\rangle^{m+1} \hat{f}(\xi) \right| d\xi.$$

In the "Taylor expansion"

$$a(\hbar) = \sum a_k \hbar^k$$

we can interpret the coefficient

$$a_k = \left(-\frac{i}{2} \right)^k \int_{\mathbb{R}^n} \langle B\xi,\xi\rangle^k \hat{f}(\xi) d\xi$$

as follows: Let $b(D)$ be the constant coefficient differential operator

$$b(D) := \sum b_{k\ell} D_k D_\ell$$

where

$$D_k = \frac{1}{i} \frac{\partial}{\partial x_k}.$$

Then $\langle B\xi,\xi\rangle^k \hat{f}(\xi)$ is the Fourier transform of the function $b(D)^k f$. So by the Fourier inversion formula,

$$(b(D)^k f)(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \langle B\xi,\xi\rangle^k \hat{f}(\xi) d\xi.$$

We can thus state our local version of the stationary phase formula as follows:

Theorem 44 *If* $f \in \mathcal{S}(\mathbb{R}^n)$ *and*

$$I(\hbar) := \int_{\mathbb{R}^n} f(x) e^{i\frac{\langle Ax,x\rangle}{2\hbar}} dx$$

then

$$I(\hbar) = \left(\frac{\hbar}{2\pi}\right)^{\frac{n}{2}} \gamma_A a(\hbar)$$

where

$$\gamma_A = |\det A|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sgn} A}$$

and $a \in C^\infty(\mathbb{R})$. *Furthermore* a *has the asymptotic expansion*

$$a(\hbar) \sim \left(\exp(-i\frac{\hbar}{2} b(D) f\right)(0).$$

The next step in our program is to use Morse's lemma.

10.6 The formula of stationary phase.

10.6.1 Critical points.

Let M be a smooth compact n-dimensional manifold, and let ψ be a smooth real valued function defined on M. Recall that a point $p \in M$ is called a *critical point* of ψ if $d\psi(p) = 0$. This means that $(X\psi)(p) = 0$ for any vector field X on M, and if X itself vanishes at p then $X\phi$ vanishes at p "to second order" in the sense that $YX\psi$ vanishes at p for any vector field Y. Thus $(YX\psi)(p)$ depends only on the value $X(p)$. Furthermore

$$(XY\psi)(p) - (YX\psi)(p) = ([X, Y]\psi)(p) = 0$$

so we get a well defined symmetric bilinear form on the tangent space TM_p called the **Hessian** of ψ at p and denoted by $d_p^2\psi$. For any pair of tangent vectors $v, w \in TM_p$ it is given by

$$d_p^2\psi(p)(v, w) := (XY\psi)(p)$$

where X and Y are any vector fields with

$$X(p) = v, \quad Y(p) = w.$$

Recall that a critical point p is called **non-degenerate** if this symmetric bilinear form is non-degenerate. We can then talk of the signature of the quadratic form $d_p^2\psi$ – i.e. the number of +'s minus the number of

-'s when we write $d_p^2\psi$ in canonical form as a sum of $\pm(x^i)^2$ where the x^i form an appropriate basis of TM_p^*. We will write this signature as $\operatorname{sgn} d_p^2\psi$ or more simply as $\operatorname{sgn}_p \psi$. The symmetric bilinear form $d_p^2\psi$ determines a symmetric bilinear form on all the exterior powers of TM_p, in particular on the highest exterior power, $\wedge^n TM_p$. This then in turn defines a density at p, assigning to every basis $v_1, \dots, v_n$ of TM_p the number

$$|d_p^2(\psi)(v_1 \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_n)|^{\frac{1}{2}}.$$

Replacing $v_1, \dots, v_n$ by $Av_1, \dots, Av_n$ has the effect of multiplying the above number by $|\det A|$ which is the defining property of a density. In particular, if we are given some other positive density at p the quotient of these two densities is a number, which we will denote by

$$|\det d_p^2\psi|^{\frac{1}{2}},$$

the second density being understood. The reason for this somewhat perverse notation is as follows: Suppose, as we always can, that we have introduced coordinates $y^1, \dots, y^n$ at p such that our second density assigns the number one to the the basis

$$v_1 = \left(\frac{\partial}{\partial y^1}\right)_p, \ \dots \ v_n = \left(\frac{\partial}{\partial y^n}\right)_p.$$

Then

$$d_p^2(\psi)(v_1 \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_n) = \det\left(\frac{\partial^2\psi}{\partial y^i \partial y^j}\right)(p)$$

so

$$|\det d_p^2\psi|^{\frac{1}{2}} = \left|\det\left(\frac{\partial^2\psi}{\partial y^i \partial y^j}\right)(p)\right|^{\frac{1}{2}}.$$

10.6.2 The formula.

With these notations let us first state a preliminary version of the formula of stationary phase. Suppose we are given a positive density, Ω, on M and that all the critical points of ψ are non-degenerate (so that there are only finitely many of them). Then for any smooth function a on M we have

$$\int_M e^{i\tau\psi} a\Omega = \left(\frac{2\pi}{\tau}\right)^{\frac{n}{2}} \sum_{p|d\psi(p)=0} e^{\frac{1}{4}\pi i \operatorname{sgn}_p \psi} \frac{e^{i\tau\psi(p)}a(p)}{|\det d_p^2\psi|^{\frac{1}{2}}} + O(\tau^{-\frac{n}{2}-1}) \tag{10.8}$$

as $\tau \to \infty$.

In fact, we can be more precise. Around every critical point we can introduce coordinates such that the Hessian of ψ is given by a quadratic form. We can also write $\Omega = b(y)dy$ for some smooth function b. We can also pull out the factor $e^{i\tau\psi(p)}$ and set $\tau^{-1} = \hbar$. We may then get the complete asymptotic expansion as given by Theorem 44.

We will prove the stationary phase formula by a series of reductions. Given any finite cover of M by coordinate neighborhoods, we may apply a partition of unity to reduce our integral to a finite sum of similar integrals, each with the function a supported in one of these neighborhoods.

By partition of unity, our proof of the stationary phase formula thus reduces to estimating integrals over Euclidean space of the form

$$\int e^{i\tau\psi(y)}a(y)dy$$

where a is a smooth function of compact support and where either

1. $d\psi \neq 0$ on supp a so that

   $$|d\psi|^2 := \left(\frac{\partial\psi}{\partial y^1}\right)^2 + \cdots + \left(\frac{\partial\psi}{\partial y^n}\right)^2 > \epsilon > 0$$

   on supp a, or

2. ψ is a non-degenerate quadratic form, which, by Sylvester's theorem in linear algebra, we may take to be of the form

   $$\psi(y) = \frac{1}{2}\left((y^1)^2 + \cdots (y^k)^2 - (y^{k+1})^2 - \cdots - (y^n)^2\right)$$

   (with, of course, the possibility that $k = 0$ in which case all the signs are negative and $k = n$ in which case all the signs are positive). The number $2k-n$ is the signature of the quadratic form ψ and is what we have denoted by $\operatorname{sgn}(d_0^2\psi)$ in the stationary phase formula.

We treat each of these two cases separately:

The case of no critical points.

In this case we will show that

$$\int e^{i\tau\psi} a dy = O(\tau^{-k}) \tag{10.9}$$

for any k.

Consider the vector field

$$X := \frac{\partial\psi}{\partial y^1}\frac{\partial}{\partial y^1} + \cdots + \frac{\partial\psi}{\partial y^n}\frac{\partial}{\partial y^n}.$$

This vector field does not vanish, and in fact

$$X\left(e^{i\tau\psi}\right) = i\tau|d\psi|^2 e^{i\tau\psi}.$$

So we can write

$$\int e^{i\tau\psi} a dy = \frac{1}{i\tau}\int X(e^{i\tau\psi})\frac{a}{|d\psi|^2}dy = \frac{1}{\tau}\int e^{i\tau\psi} b dy$$

where

$$b = iX\left(\frac{a}{|d\psi|^2}\right)$$

by integration by parts. Repeating this integration by parts argument proves(10.9). This takes care of the case where there are no critical points.

The case near a critical point.

We assume that p is an isolated critical point, and we have chosen coordinates y about p such that p has coordinates $y = 0$ and that $\psi = \psi(p) + \frac{1}{2}Q(y)$ in these coordinates where $Q(y)$ is a diagonal quadratic form. We now have a single summand on the right of (10.8) and by pulling out the factor $e^{i\tau\psi(p)}$ we may assume that $\psi(p) = 0$. Now apply Theorem 44. □

We now turn to various applications of the formula of stationary phase.

10.7 Group velocity.

In this section we describe one of the most important applications of stationary phase to physics. Let $\hbar$ be a small number (eventually we will take $\hbar = h/2\pi$ where h is Planck's constant, but for the moment we want to think of $\hbar$ as a parameter which approaches zero, so that $\tau := (1/\hbar) \to \infty$). We want to consider a family of "traveling waves"

$$e^{-(i/\hbar)(E(p)t - p\cdot x)}.$$

For simplicity in exposition we will take p and x to be scalars, but the discussion works as well for x a vector in three (or any) dimensional space and p a vector in the dual space. For each such wave, and for each fixed time t, the wave number of the space variation is h/p. Since we allow E to depend on p, each of these waves will be traveling with a possibly different velocity. Suppose we superimpose a family of such waves, i.e. consider an integral of the form

$$\int a(p) e^{-(i/\hbar)(E(p)t - p\cdot x)} dp. \tag{10.10}$$

Furthermore, let us assume that the function $a(p)$ has its support in some neighborhood of a fixed value, p_0. Stationary phase says that the only non-negligible contributions to the above integral will come from values of p for which the derivative of the exponent with respect to p vanishes, i.e. for which

$$E'(p)t - x = 0.$$

Since $a(p)$ vanishes unless p is close to p_0, this equation is really a constraint on x and t. It says that the integral is essentially zero except for those values of x and t such that

$$x = E'(p_0)t \tag{10.11}$$

holds approximately. In other words, the integral looks like a little blip called a *wavepacket* when thought of as a function of x, and this blip moves with velocity $E'(p_0)$ called the *group velocity*.

Let us examine what kind of function E can be of p if we demand invariance under (the two dimensional version of) all Lorentz transformations, which are all linear transformations preserving the quadratic form $c^2t^2 - x^2$. Since (E, p) lies in the dual space to (t, x), the dual Lorentz transformation sends $(E, p) \mapsto (E', p')$ where

$$E^2 - c^2p^2 = (E')^2 - c^2(p')^2$$

and given any (E, p) and (E', p') satisfying this condition, we can find a Lorentz transformation which sends one into the other. Thus the only invariant relation between E and p is of the form

$$E^2 - (pc)^2 = \text{constant}.$$

Let us call this constant m^2c^4 so that $E^2 - (pc)^2 = m^2c^4$ or

$$E(p) = \left((pc)^2 + m^2c^4\right)^{1/2}.$$

Then

$$E'(p) = \frac{pc^2}{E(p)} = \frac{p}{M}$$

where M is defined by

$$E(p) = Mc^2 \quad \text{or} \quad M = \left(m^2 + \left(\frac{p}{c}\right)^2\right)^{1/2}.$$

Notice that if p/c is small in comparison with m then $M \doteq m$. If we think of M as a *mass*, then the relationship between the group velocity $E'(p)$ and p is precisely the relationship between velocity and momentum in classical mechanics. In this way have associated a wave number $k = p/h$ to the momentum p and if we think of E as energy we have associated the frequency $\nu = E/h$ to energy. We have established the three famous formulas

$$\begin{array}{lllll} E & = & c^2\left(m^2 + \left(\frac{p}{c}\right)^2\right)^{1/2} & \doteq & mc^2 & \text{Einstein's mass energy formula} \\ \lambda & = & \frac{1}{k} & = & \frac{h}{p} & \text{de Broglie's formula} \\ E & = & h\nu & & & \text{Einstein's energy frequency formula.} \end{array}$$

In these formulas we have been thinking of h or $\hbar$ as a small parameter tending to zero. The great discovery of quantum mechanics is that h should not tend to zero but is a fundamental constant of nature known as *Planck's constant*. In the energy frequency formula it occurs as a conversion factor from inverse time to energy, and hence has units energy $\times$ time. It is given by

$$h = 6.626 \times 10^{-34}\text{J s}.$$

10.8 The Fourier inversion formula.

We used the Fourier transform and the Fourier inversion formula to derive the lemma of stationary phase. But if we knew stationary phase then we could derive the fourier inversion formula as follows:

Consider the function $p = p(x, \xi)$ on $\mathbb{R}^n \oplus \mathbb{R}^n$ given by

$$p(x, \xi) = x \cdot (\xi - \eta)$$

where $\eta \in \mathbf{R}^n$. This function has only one critical point, at

$$x = 0, \xi = \eta$$

where its signature is zero. We conclude that for any such function $a = a(x, \xi) \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n$ we have

$$\int\int e^{i\tau x\cdot(\xi-\eta)} a(x,\xi) dx d\xi = \left(\frac{2\pi}{\tau}\right)^n a(0,\eta) + O(\tau^{-(n+1)}).$$

Let us choose $a(x, \xi) = f(x)g(\xi)$ where f and g are smooth functions vanishing rapidly with their derivatives at infinity. We get

$$\left(\frac{1}{\tau^n}\right) f(0)g(\eta) = \frac{1}{(2\pi)^n} \int\int e^{i\tau x\cdot(\xi-\eta)} f(x)g(\xi) dx d\xi + O(\tau^{-(n+1)}).$$

Let us set $u = \tau x$ in the integral, so that $dx = \tau^{-n} du$. Multiplying by τ^n we get

$$f(0)g(\eta) = \frac{1}{(2\pi)^n} \int\int f\left(\frac{u}{\tau}\right) g(\xi) e^{iu\cdot(\xi-\eta)} du d\xi + O(\tau^{-1}).$$

So if we define

$$\hat{g}(u) := \frac{1}{(2\pi)^{n/2}} \int g(\xi) e^{i\xi\cdot u} d\xi$$

we have proved that

$$f(0)g(\eta) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} f\left(\frac{u}{\tau}\right) \hat{g}(u) e^{iu\cdot\eta} du + O(\tau^{-1}).$$

If we choose f such that $f(0) = 1$ and let $\tau \to \infty$ we obtain the Fourier inversion formula:

$$g(\eta) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \hat{g}(u) e^{iu\cdot\eta} du.$$

10.9 Fresnel's version of Huygen's principle.

10.9.1 The wave equation in one space dimension.

As a warm up to the study of spherical waves in three dimensions we study the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

where $u = u(x,t)$ with x and t are real variables.

If we make the change of variables $p = x + t, q = x - t$ this equation becomes

$$\frac{\partial^2 u}{\partial p \partial q} = 0$$

and so by integration

$$u = u_1(p) + u_2(q)$$

where u_1 and u_2 are arbitrary differentiable functions. Reverting to the original coordinates this becomes

$$u(x,t) = u_1(x+t) + u_2(x-t). \tag{10.12}$$

Any such function is clearly a solution. The function $u_2(x-t)$ can be thought of dynamically: At each instant of time t, the graph of $x \mapsto u_2(x-t)$ is given by the graph of $x \mapsto u_2(x)$ displaced t units to the right. We say that $u_2(x-t)$ represents a **traveling wave** moving without distortion to the right with unit speed.

Thus the most general solution of the homogeneous wave equation in one space dimension is given by the superposition of two undistorted traveling wave, one moving to the right and the other moving to the left.

10.9.2 Spherical waves in three dimensions.

In three space dimensions the wave equation (in spherical coordinates) is

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

If $u = u(r,t)$ the last two terms on the right disappear while

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial u}{\partial r} = \frac{1}{r}\left[2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right] = \frac{1}{r}\frac{\partial^2 (ru)}{\partial r^2}.$$

Thus $v := ru$ satisfies the wave equation in one space variable, and so the general spherically symmetric solution of the wave equation in three space dimensions is given by

$$u(r,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}.$$

The first term represents and incoming spherical wave and the second term an outgoing spherical wave. In particular, if we take $f = 0$ and $g(s) = e^{iks}$ then

$$w_k(r,t) := \frac{e^{ik(r-t)}}{r}$$

is an outgoing spherical sinusoidal wave of frequency k.

10.9.3 Helmholtz's formula

Recall Green's second formula (a consequence of Stokes' formula) which says that if u and v are smooth functions on a bounded region $V \subset \mathbb{R}^3$ with piecewise smooth boundary ∂V then

$$\int_{\partial V}(u \star dv - v \star du) = \int_V (v\Delta u - u\Delta v) dx \wedge dy \wedge dz.$$

In particular, if u and v are both solutions of the reduced wave equation $\Delta\phi - k^2\phi = 0$ the right hand side vanishes, and we get

$$\int_{\partial V}(u \star dv - v \star du) = 0.$$

Now

$$d\left(\frac{e^{ikr}}{r}\right) = \frac{e^{ikr}}{r}\left[ik - \frac{1}{r}\right] dr. \qquad (10.13)$$

Let D be a bounded domain with piecewise smooth boundary, let r_P denote the distance from a point P interior to D, and take V to consist of those points of D exterior to a small sphere about P. Then if v is a solution to the reduced wave equation and we take $u = e^{ikr}/r$ we obtain Helmhotz's formula

$$v(P) = \frac{1}{4\pi}\int_{\partial D}\left[\frac{e^{ikr_P}}{r_P}\star dv - v\star d\frac{e^{ikr_P}}{r_P}\right]$$

by shrinking the small sphere to zero.

Green's formula also implies that if P is exterior to D the integral on the right vanishes.

Now let D consist of all points exterior to a surface S but inside a ball of radius R centered at P. If Σ_R denotes the sphere of radius R centered at P, then the contribution to Helmhotz's formula coming from integrating over σ_R will be the integral over the unit sphere

$$\int e^{ikr}\left[r\left(\frac{\partial v}{\partial r} - iku\right) + v\right]\Big|_{r=R} d\omega$$

where $d\omega$ is the area element of the unit sphere. This contribution will go to zero if the **Sommerfeld radiation conditions**

$$\int |v| d\omega = o(1), \quad \text{and} \qquad \int \left|\frac{\partial v}{\partial r} - iku\right| d\omega = o(R^{-1})$$

are satisfied (where the integrals are evaluated at $r = R$).

Assuming these conditions, we see that if P is exterior to S then

$$v(P) = \frac{1}{4\pi}\int_S\left[\frac{e^{ikr_P}}{r_P}\star dv - v\star d\frac{e^{ikr_P}}{r_P}\right]. \quad (10.14)$$

while the integral vanishes if P is inside S.

Huyghens had the idea that propagated disturbances in wave theory could be represented as the superposition of secondary disturbances along an intermediate surface such as S. But he did not have an adequate explanation as to why there was no "backward wave", i.e. why the propagation was only in the outward direction. Fresnel believed that if all the original sources of radiation were inside S, the integrand in Helmholtz's formula would vanish due to interference. The above argument due to Helmholtz was the first rigorous mathematical treatment of the problem, and shows that the internal cancellation is due to the total effect of the boundary.

However, we will see, by using stationary phase, that Fresnel was right up to order $1/k$.

10.9.4 Asymptotic evaluation of Helmholtz's formula

We will assume that near S the v that enters into (10.14) has the form

$$v = ae^{ik\phi}$$

where a and ϕ are smooth and $\|\operatorname{grad}\phi\| \equiv 1$. For example, if v represent radiation from some point Q interior to S then this would hold with $\phi = r_Q$.

We assume that P is sufficiently far from S so that $1/r_P$ is negligible in comparison with k, and we also assume that a and da are negligible in comparison with k. As P will be held fixed, we will write r for r_P. Then inserting (10.13) into (10.14) shows that the leading term in(10.14) (in powers of k) is

$$\frac{ik}{4\pi}\int_S \frac{a}{r}e^{ik(\phi+r)}(\star d\phi - \star dr).$$

We want to apply stationary phase to this integral. The critical points are those points y on S at which the restriction of $d\phi+dr$ to S vanishes. This says that the projection of $\operatorname{grad}\phi(y)$ onto the tangent space to S at y is the negative of the projection of $\operatorname{grad} r(y)$ onto this tangent space. Since $\|\operatorname{grad}\phi\| = \|\operatorname{grad} r\| = 1$, this implies that the projections of $\operatorname{grad}\phi(y)$ and $\operatorname{grad} r(y)$ onto the normal have the same absolute value. There are thus two possibilities:

1. $\operatorname{grad}\phi(y) = -\operatorname{grad} r(y)$. In this case $\star d\phi(y) = -\star dr(y)$ when restricted to the tangent space to S at y.

2. $\operatorname{grad}\phi(y) = 2(\operatorname{grad}\phi(y), n)n - \operatorname{grad} r(y)$. In this case $\star d\phi(y) = \star dr(y)$ when restricted to the tangent space to S at y.

Let us assume for the moment that the critical points are non-degenerate. (We will discuss this condition below.)

Suppose we are in case 2). Then the leading term in the integral in (10.14) vanishes, and hence the contribution from (10.14) is of order $1/k$. If S wre convex and $\operatorname{grad}\phi$ pointed outward, then for any P insised S we would be in case 2). This justifies Fresnel's view that there is local cancellation of the backward wave (at least up to terms of order $1/k$).

10.9.5 Fresnel's hypotheses.

Suppose we are in case 1). Then the leading term in (10.14) is

$$\frac{ik}{4\pi}\int_S \frac{a}{r}e^{ik(\phi+r)} \star dr.$$

This shows that up to terms of order $1/k$ the "induced secondary radiation" coming from S behaves as if it

- has amplitude equal to $1/\lambda$ times the amplitude of the primary wave where $\lambda = 2\pi/k$ is the wave length, and

- has phas one quarter of a period ahead of the primary wave. (This is one way of interpreting the factor i.)

Fresnel made these two assumptions directly in his formulation of Huyghen's principle leading many to regard them as *ad hoc*. We see that it is a consequence of Helmholtz's formula and stationary phase.

10.10 The lattice point problem.

Let D be a domain in the plane with piecewise smooth boundary. The high school method of computing the area of D is to superimpose a square grid on the plane and count the number of squares "associated" with D. Since some squares may intersect D but not be contained in D, we must make a choice: let us choose to count all squares which intersect $\overline{D}$. Furthermore, in order to avoid unnecessary notation, let us assume that D is taken to include its boundary, i.e. D is closed: $D = \overline{D}$. If we let $\mathbf{Z}^2$ denote the lattice determined by the corners of our grid, then our procedure is to count the number of points in

$$D \cap \mathbf{Z}^2.$$

Of course this is only an approximation to the area of D. To get better and better approximations we would shrink the size of the grid. Our problem is to find an estimate for the error in this procedure.

For notational reasons, it is convenient to keep the lattice fixed, and dilate the domain D. That is, we want to count the number of lattice points in λD where λ is a (large) positive real number. So we set

$$N_D^\sharp(\lambda) := \#(\lambda D \cap \mathbf{Z}^2). \tag{10.15}$$

Equally well, if χ^D denotes the indicator function (sometimes called the characteristic function) of D:

$$\chi^D(x) = 1 \text{ if } x \in D, \qquad \chi^D(x) = 0 \text{ if } x \notin D,$$

then

$$N_D^\sharp(\lambda) = \sum_{\nu \in \mathbf{Z}^2} \chi_\lambda^D(\nu), \tag{10.16}$$

where

$$\chi_\lambda(x) := \chi\left(\frac{x}{\lambda}\right).$$

(Frequently, in what follows, we will drop the D when D is fixed. Also, we will pass from 2 to n with the obvious minor changes in notation.)

Now it is clear that

$$N_D^\sharp(\lambda) = \lambda^2 \cdot \text{Area}(D) + \text{ error}.$$

Our problem is to estimate the error. Without any further assumptions, it is relatively easy to see that we can certainly say that the error can be estimated by a constant times λ where the constant involves only the length of ∂D. In general, we can not do better, especially if the boundary of D contains straight line segments of rational slope: For the worst possible scenario, consider the case where D is a square centered at the origin. Then every time that λ is such that the vertices of λD lie in $\mathbf{Z}^2$, then the number of boundary points lying in $\mathbf{Z}^2$ will be proportional to λ times the length of the perimeter of D. But a slightly larger or small value of λ will yield no boundary points in $\mathbf{Z}^2$. We might expect that if the boundary is curved everywhere, we can improve on the estimate of the error.

The main result of this section, due to Van der Corput, asserts that if D is convex, with smooth boundary whose curvature is everywhere positive (we will give more precise definitions later) then we can estimate the error terms as being

$$O(\lambda^{\frac{2}{3}}).$$

In fact, Van der Corput shows that this result is sharp if we allow all such strongly convex smooth domains, although we will not establish this result here.

10.10.1 The circle problem.

Suppose that we take D to be the unit disk. In this case

$$N_D^\sharp(\lambda) = N(\lambda)$$

where

$$N(\lambda) = \#\{\nu = (m,n) \in \mathbf{Z}^2 | m^2 + n^2 \leq \lambda^2\}. \quad (10.17)$$

In this case, there will only be lattice points on the boundary of λD if λ^2 is an integer which can be represented as a sum of two squares, and the number of points on the boundary will be the number of ways of representing λ^2 as a sum of two squares.

The number of ways of representing an integer N as the sum of two integer squares is closely related to the number of prime factors of N of the form $4k+1$ and the number of prime square factors of the form $4k+3$. In fact, as we shall remind you later on, if $r(N)$ denotes the number of ways of writing N as a sum of two squares then $r(N)$ can be evaluated as follows: Suppose we factorize N into prime powers, collect all the powers of 2, collect all the primes congruent to 1 (mod 4), and collect all the primes which are congruent to 3 (mod 4). In other words, we write

$$N = 2^f N_1 N_2 \quad (10.18)$$

where

$$N_1 = \prod p^r \qquad p \equiv 1 (\text{mod } 4)$$

and

$$N_2 = \prod q^s \qquad q \equiv 3 (\text{mod } 4).$$

Then $r(N) = 0$ if any s is odd. If all the s are even, then

$$r(N) = 4d(N_1). \quad (10.19)$$

So there are relatively few points on the boundary of λD when D is the unit disk, and we might expect special results in this case. Of course our problem is to estimate the number of lattice points close to a given circle, not necessarily exactly on it.

Let us set

$$t := \lambda^2, \quad (10.20)$$

as the square of λ is the parameter used frequently in the number theoretical literature. Let us define $R(t)$

as the error in terms of t, so

$$\sum_{n\leq t} r(n) = \pi t + R(t). \tag{10.21}$$

Then the result of Van der Corput cited above asserts that

$$R(t) = O(t^{\frac{1}{3}}). \tag{10.22}$$

In fact, later work of Van der Corput himself in the twenties and early thirties, involving the theory of "exponent pairs" improves upon this estimate. For example, one consequence of the method of "exponent pairs" is that

$$R(t) = O(t^{\frac{27}{82}}). \tag{10.23}$$

In fact, the long standing conjecture (going back to Gauss, I believe) has been that

$$R(t) = O(t^{\frac{1}{4}+\epsilon}) \quad \text{for any} \quad \epsilon > 0. \tag{10.24}$$

Notice the sequence of more and more refined results: trivial arguments, valid for any region with piecewise smooth boundary give an estimate $R(t) = O(t^\rho)$ where $\rho = \frac{1}{2}$. The Van der Corput method valid for all smooth strongly convex domains gives $\rho = \frac{1}{3}$. The method of exponent pairs gives $\rho = (k+\ell)/(2k+2)$ whenever (k,ℓ) is an exponent pair, but although this method improved on $\frac{1}{3}$, it did not yield the desired conjecture - that we may take $\rho = \frac{1}{4}+\epsilon$ for any $\epsilon > 0$.

10.10.2 The divisor problem.

Let $d(n)$ denote the number of divisors of the positive integer n. Using elementary arguments, Dirichlet (1849) showed that

$$\sum_{n\leq t} d(n) = t(\log t + 2\gamma - 1) + O(t^{\frac{1}{2}}) \tag{10.25}$$

where γ is Euler's constant

$$\gamma := \lim_{N\to\infty}\left(\sum_{n\leq N}\frac{1}{n} - \log N\right).$$

Dirichlet's argument is as follows: First of all observe that we can regard the divisor problem as a lattice point counting problem. Indeed, consider the region, T_t, in the (x,y) plane bounded by the hyperbola $xy = t$ and the straight line segments from $(1,1)$ to $(1,t)$ and from $(1,1)$ to $(t,1)$. So T_t is a "triangle" with the hypotenuse replaced by a hyperbola. Then $d(n)$ is the number of lattice points on the "integer hyperbola" $xy = n, \ n \le t$, and so $\sum_{n\le t} d(n)$ is the total number of lattice points in T_t. The area of T_t is $t\log t - t + 1$, which has the same leading term as above. To count the number of lattice points in T_t, observe that T_t is symmetric about the line $y = x$, and there are $[\sqrt{t}]$ lattice points in T_t on this line. For each integer $d \le [\sqrt{t}]$ the number of lattice points on the horizontal line) $y = d$ in T_t to the right of the diagonal is

$$\left[\frac{t}{d}\right] - d$$

so

$$\sum_{n\le t} d(n) = 2\sum_{d\le\sqrt{t}}\left(\left[\frac{t}{d}\right] - d\right) + \left[\sqrt{t}\right].$$

Since $[s] = s + O(1)$ we can write this as

$$2t\sum_{d\le\sqrt{t}}\frac{1}{d} - 2\cdot\frac{\sqrt{t}(\sqrt{t}+1)}{2} + O(\sqrt{t}).$$

The formula leading to Euler's constant has error term $1/s$:

$$\sum_{n\le s}\frac{1}{d} = \log s + \gamma + O(\frac{1}{s}) \tag{10.26}$$

as follows from Euler MacLaurin (see later on). So setting $s = \sqrt{t}$ in the above we get (10.25).

Once again we may ask if this estimate can be improved: Define

$$\Delta(t) := \sum_{n\le t} d(n) - t(\log t + 2\gamma - 1) \tag{10.27}$$

and ask for better σ such that

$$\Delta(t) = O(t^\sigma) \tag{10.28}$$

It turns out, that the method of exponent pairs yields the same answer as in the circle problem case: If (k,ℓ) is an "exponent pair" then

$$\sigma = (k+\ell)/(2k+2)$$

is a suitable exponent in (10.28). Once again, the conjectured theorem has been that we may take $\sigma = \frac{1}{4} + \epsilon$ for any positive ϵ.

These "lattice point problems" are closely related to studying the growth of the Riemann zeta function on the critical line, i.e. to obtain power estimates for $\zeta(\frac{1}{2} + it)$. Furthermore, the Riemann hypothesis itself is known to be closely related to somewhat deeper "approximation" problems. See, for example, the book *Area, Lattice Points, and Exponential Sums* by M.N Huxley, page 15.

10.10.3 Using stationary phase.

Van der Corput revolutionized the study of the lattice point problem in the 1920's by bringing to bear two classical tools of analysis - the Poisson summation formula and the method of stationary phase.

Our application will be of the following nature: Recall that a subset of $\mathbf{R}^n$ is convex if it is the intersection of all the half spaces containing it. Suppose that D is a (compact) convex domain with smooth boundary, containing the origin and that u is a unit vector. Then the function $y \mapsto u \cdot y$ achieves a maximum m^+ and a minimum m^- on D and the condition that these be taken on at exactly one point each is what is usually meant by saying that D is strictly convex. We want to impose the stronger condition that restriction of the function $y \mapsto u \cdot y$ to the boundary is non-degenerate having only two critical points, the maximum and the minimum, for all unit vectors. This has the following consequence: Let K be a compact subset of $\mathbf{R}^n - \{0\}$ and consider the Fourier transform of the indicator function $\chi = \chi^D$ evaluated at τx for $x \in K$:

$$\hat{\chi}(\tau x) = \int_D e^{i\tau x \cdot y} dy.$$

(For today it will be convenient to use this definition of the Fourier transform so that

$$\hat{\chi}(0) = \text{vol } (D)$$

without the factors of 2π.)

Holding x fixed, we have (as differential forms in y)

$$d\left(e^{i\tau x\cdot y} x^1 dy^2 \wedge \cdots \wedge dy^n\right) = i\tau (x^1)^2 e^{i\tau x \cdot y} dy^1 \wedge \cdots dy^n$$

so

$$e^{i\tau x\cdot y}dy = e^{i\tau x\cdot y}dy^1 \wedge \cdots \wedge dy^n = \frac{1}{i\tau|x|^2}d\left(e^{i\tau x\cdot y}\omega\right)$$

where

$$\omega := x^1 dy^2\wedge\cdots\wedge dy^n - x^2 dy^1\wedge dy^3 \cdots\wedge dy^n + \cdots \pm x^n dy^1\wedge\cdots\wedge dy^{n-1}.$$

By Stokes,

$$\hat{\chi}(\tau x) = \frac{1}{i\tau|x|^2}\int_{\partial D} e^{i\tau x\cdot y}\omega.$$

The integral on the right is $O(\tau^{-\frac{n-1}{2}})$ by stationary phase, and hence

$$\hat{\chi}(\tau x) = O(\tau^{-\frac{n+1}{2}}) \tag{10.29}$$

uniformly for $x \in K$ where K is any compact subset of $\mathbf{R}^n - \{0\}$. As this is the property we will use, we might as well take this as the definition of a **strongly convex region**.

10.10.4 Recalling Poisson summation.

The second theorem from classical analysis that goes into the proof of Van der Corput's theorem is the Poisson summation formula. This says that if f is a smooth function vanishing rapidly with its derivatives at infinity on $\mathbf{R}^n$ then (in the current notation)

$$\sum_{\mu\in\mathbf{Z}^n} \hat{f}(2\pi\mu) = \sum_{\nu\in\mathbf{Z}^n} f(\nu). \tag{10.30}$$

We recall the elementary proof of this fact :
Set

$$h(x) := \sum_{\nu\in\mathbf{Z}^n} f(x+\nu)$$

so that h is a smooth periodic function with period the unit lattice, $\mathbf{Z}^n$. By definition

$$h(0) = \sum_{\nu\in\mathbf{Z}^n} f(\nu).$$

Since h is periodic, we may expand it into a Fourier series

$$h(x) = \sum_{\mu\in\mathbf{Z}^n} c_\mu e^{-2\pi i\mu\cdot x}$$

where

$$c_\mu = \int_0^1 \cdots \int_0^1 h(x)e^{2\pi i\mu\cdot x}dx = \int_0^1 \cdots \int_0^1 \sum_{\nu\in\mathbf{Z}^n} f(x+\nu)e^{2\pi i\mu\cdot x}dx.$$

We may interchange the order of summation and integration and make the change of variables $x+\nu \mapsto x$ to obtain

$$c_\mu = \hat{f}(2\pi\mu).$$

Setting $x = 0$ in the Fourier series

$$h(x) = \sum_{\mu\in\mathbf{Z}^n} \hat{f}(2\pi\mu)e^{-2\pi i\mu\cdot x}$$

gives

$$h(0) = \sum_{\mu\in\mathbf{Z}^n} \hat{f}(2\pi\mu).$$

Equating the two expressions for $h(0)$ is (10.30).

10.11 Van der Corput's theorem.

In n-dimensions this says:

Theorem 45 *Let D be a strongly convex domain. Then*

$$N_D^\sharp(\lambda) = \lambda^n \operatorname{vol}(D) + O(\lambda^{n-2+\frac{2}{n+1}}) \qquad (10.31)$$

Proof. Let $\chi = \chi^D$ be the indicator function of D so that χ_λ defined by

$$\chi_\lambda(y) := \chi(\frac{y}{\lambda})$$

is the indicator (characteristic) function of λD. Thus

$$N^\sharp(\lambda) = \sum_{\nu\in\mathbf{Z}^n} \chi_\lambda(\nu)$$

where we have written $N^\sharp$ for $N_D^\sharp$. The Fourier transform of χ_λ is given in terms of the Fourier transform of χ by

$$\hat{\chi_\lambda}(x) = \lambda^n\hat{\chi}(\lambda x).$$

Furthermore,

$$\hat{\chi}(0) = \operatorname{vol}(D).$$

If we could apply the Poisson summation formula directly to χ_λ then the contribution from 0 would be

$\lambda^n \operatorname{vol}(D)$, and we might hope to control the other terms using (10.29). (For example, if we could brutally apply (10.29) to control *all* the remaining terms in the case of the circle, we would be able to estimate the error in the circle problem as $\lambda^{2-3/2} = \lambda^{1/2}$ which is the circle conjecture.) But this will not work directly since χ_λ is not smooth. We must first regularize χ_λ and the clever idea will be to choose this regularization to depend the right way on λ.

So let ρ be a non-negative smooth function on $\mathbf{R}^n$ supported in the unit ball with integral one. Let

$$\rho_\epsilon(y) = \frac{1}{\epsilon^n}\rho\left(\frac{y}{\epsilon}\right)$$

so ρ_ϵ is supported in the ball of radius ϵ and has total integral one. Thus

$$\hat{\rho}_\epsilon(x) = \hat{\rho}(\epsilon x)$$

and

$$\hat{\rho}(0) = 1.$$

Define

$$N^\sharp_\epsilon(\lambda) = \sum_{\nu\in\mathbf{Z}^n}(\chi_\lambda \star \rho_\epsilon)(\nu)$$

where $\star$ denotes convolution. If ν lies a distance greater than ϵ from the boundary of λD, then $(\chi_\lambda \star \rho_\epsilon)(\nu) = \chi_\lambda(\nu)$. Thus

$$N^\sharp_\epsilon(\lambda - C\epsilon) \le N^\sharp(\lambda) \le N^\sharp_\epsilon(\lambda + C\epsilon)$$

where C is some constant depending only on D. Suppose we could prove that $N^\sharp_\epsilon$ satisfies an estimate of the type (10.31). Then we could conclude that

$$(\lambda - C\epsilon)^n \operatorname{vol}(D) + O(\lambda^{n-2+\frac{2}{n+1}}) \le N^\sharp(\lambda) \le (\lambda + C\epsilon)^n + O(\lambda^{n-2+\frac{2}{n+1}}).$$

Suppose we set

$$\epsilon = \lambda^{-1+\frac{2}{n+1}}. \tag{10.32}$$

Then

$$(\lambda \pm C\epsilon)^n = \lambda^n + O(\lambda^{n-2+\frac{2}{n+1}})$$

and we obtain the Van der Corput estimate for $N^\sharp(\lambda)$. So it is enough to prove the analogue of (10.31) with $N^\sharp_\epsilon$ watching out for the dependence on ϵ.

Since $\chi_\lambda \star \rho_\epsilon$ is smooth and of compact support, and since

$$(\chi_\lambda \star \rho_\epsilon)\hat{} = \hat{\chi}_\lambda \cdot \hat{\rho}_\epsilon$$

we may apply the Poisson summation formula to conclude that

$$N_\epsilon^\sharp(\lambda) = \lambda^n \operatorname{vol}(D) + \sum_{\nu \in \mathbf{Z^n} - \{\mathbf{0}\}} \lambda^n \hat{\chi}(2\pi\lambda\nu)\hat{\rho}(2\pi\epsilon\nu)$$

and we must estimate the sum on the right hand side. Now since ρ is of compact support its Fourier transform vanishes faster than any inverse power of $(1+|x|^2)$. So, using (10.29) we can estimate this sum by

$$\lambda^{n-\frac{n+1}{2}} \sum_{\nu \in \mathbf{Z}^n - \{0\}} |\nu|^{-\frac{n+1}{2}}(1+|\epsilon\nu|^2)^{-K}$$

were K is large, or, what is the same by

$$\lambda^{\frac{n-1}{2}} \int \frac{1}{|x|^{\frac{n+1}{2}}}(1+|\epsilon x|^2)^{-K} dx$$

where K is large. Making the change of variables $x = \epsilon z$ this becomes

$$\lambda^{\frac{n-1}{2}} \epsilon^{-\frac{n-1}{2}} \int \frac{1}{|z|^{\frac{n+1}{2}}}(1+|z|)^{-K} dz.$$

The integral does not depend on anything, and if we substitute (10.32) for ϵ, the power of λ that we obtain is

$$\frac{n-1}{2} - \frac{n-1}{2}\left(-1 + \frac{2}{n+1}\right) = \frac{n-1}{2} + \frac{n-1}{2} - \frac{n+1}{n+1} + \frac{2}{n+1} = n-2+\frac{2}{n+1}$$

proving (10.31). $\square$